Clay hypoplasticity model including stiffness anisotropy
David Mašín

correspondence address:
Charles University in Prague
Faculty of Science
Albertov 6
12843 Prague 2, Czech Republic
E-mail: masin@natur.cuni.cz
Tel: +420-2-2195 1552, Fax: +420-2-2195 1556

Submitted for publication in Géotechnique
Article type: Technical Note
Number of words main text: 1844
Number of words other parts: 607 (references), 289 (appendix), 83 (abstract).
Nov 26 2013

Abstract

Hypoplastic model for clays is developed predicting anisotropy of very small strain stiffness. The existing hypoplastic model with explicit formulation of the asymptotic state boundary surface is combined with an anisotropic form of the stiffness tensor. Naturally, the resultant model predicts correctly the very small strain stiffness anisotropy. It is demonstrated that properly are also predicted trends in the anisotropy influence on undrained stress paths. The model is evaluated using hollow cylinder apparatus experimental data on London clay taken over from literature.

Keywords: clays; anisotropy; constitutive relations; stiffness; stress path

Introduction

Anisotropy of sedimentary clays is such a significant feature of their mechanical behaviour that it cannot be ignored in boundary value problem simulations. For example, Addenbrooke et al. (1997), Gunn (1993), Ng et al. (2004) and Franzius et al. (2005) demonstrated that incorporation of stiffness anisotropy improved predictions of tunnelling problems. In this Note, we develop a hypoplastic model for clays incorporating very small strain stiffness anisotropy. Like the underlining hypoplastic models, the model is capable of predicting small strain stiffness non-linearity, recent stress history effects (Atkinson et al. 1990) and large-strain asymptotic behaviour (Gudehus and Mašín 2009, Mašín 2012a). The model is based on the earlier research by the author, which will only briefly be summarised here due to limited space. For details of hypoplastic modelling, the readers are referred to the cited publications.
Hypoplasticity is an approach to non-linear constitutive modelling of geomaterials. In its general form by (Gudehus 1996) it may be written as

\[ \dot{\sigma} = f_s(\mathbf{L} : \dot{\epsilon} + f_d \mathbf{N} \|\dot{\epsilon}\|) \]  

(1)

where \( \dot{\sigma} \) and \( \dot{\epsilon} \) represent the objective (Zaremba-Jaumann) stress rate and the Euler stretching tensor respectively, \( \mathbf{L} \) and \( \mathbf{N} \) are fourth- and second-order constitutive tensors, and \( f_s \) and \( f_d \) are two scalar factors. In hypoplasticity, stiffness predicted by the model is controlled by the tensor \( \mathbf{L} \), while strength (and asymptotic response in general, see Mašín (2012a), is governed by a combination of \( \mathbf{L} \) and \( \mathbf{N} \). Earlier hypoplastic models (such as the model by Wolffersdorff 1996 and Mašín 2005) did not allow to change the \( \mathbf{L} \) formulation arbitrarily, as any modification of the tensor \( \mathbf{L} \) undesirably influenced the predicted asymptotic states. This hypoplasticity limitation was overcome by Mašín (2012c). He developed an approach enabling to specify the asymptotic state boundary surface independently of the tensor \( \mathbf{L} \) and demonstrated it by proposing a simple hypoplastic equivalent of the Modified Cam-clay model. Based this approach, Mašín (2012b) developed an advanced hypoplastic model for clays. This model will serve as a base model for current developments.

**Model formulation**

The model presented in this Note combines hypoplastic model from Mašín (2012b) with anisotropic form of the tensor \( \mathbf{L} \) proposed by Mašín and Rott (2013). Mašín and Rott (2013) adopted general transversely elastic model formulation, which reads (Spencer 1982, Lubarda and Chen 2008)

\[ \mathbf{L} = \frac{1}{2}a_1 \mathbf{1} \otimes \mathbf{1} + a_2 \mathbf{1} \otimes a_3 (\mathbf{p} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{p}) + a_4 \mathbf{p} \otimes \mathbf{1} + a_5 \mathbf{p} \otimes \mathbf{p} \]  

(2)

where the tensor products represented by "\( \otimes \)" and "\( \circ \)" are defined as

\[ (\mathbf{p} \otimes \mathbf{1})_{ijkl} = p_{ij} 1_{kl} \]

\[ (\mathbf{p} \circ \mathbf{1})_{ijkl} = \frac{1}{2}(p_{ik} 1_{jl} + p_{jl} 1_{ik} + p_{jl} 1_{ik} + p_{jk} 1_{il}) \]  

(3)

where \( p_{ij} = n_i n_j \); \( n_i \) is a unit vector normal to the plane of symmetry (in sedimentary soils this vector typically represents the vertical direction). \( a_1 \) to \( a_5 \) in Eq. (2) represent five material constants. They can be calculated as

\[ a_1 = \alpha_E \left( 1 - v_{pp} - \frac{2 \alpha_E v_{pp}^2}{\alpha_v^2} \right) \]  

(4)

\[ a_2 = \alpha_E v_{pp} \left( 1 + \frac{\alpha_E}{\alpha_v} v_{pp} \right) \]  

(5)

\[ a_3 = \alpha_E v_{pp} \left( \frac{1}{\alpha_v} + \frac{v_{pp}}{\alpha_v} - 1 - \frac{\alpha_E}{\alpha_v} v_{pp} \right) \]  

(6)

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1 Similar problem has already been discussed by Kopito and Klar (2013), who incorporated transversely isotropic stiffness tensor \( \mathbf{L} \) into the model by Mašín (2012c).
where the anisotropy coefficients $\alpha_G$, $\alpha_E$ and $\alpha_v$ are defined as

\begin{align}
\alpha_G &= \frac{G_{pp}}{G_{tp}} \\
\alpha_E &= \frac{E_p}{E_t} \\
\alpha_v &= \frac{\nu_{pp}}{\nu_{tp}}
\end{align}

$G_{ij}$ are shear moduli, $E_i$ are Young moduli and $\nu_{ij}$ are Poisson ratios. Subscript "p" denotes direction within the plane of isotropy (typically horizontal direction) and subscript "t" denotes direction transverse to the plane of isotropy (typically vertical direction).

When compared to the reference model, incorporation of anisotropic form of $\mathbf{L}$ requires re-evaluation of the factor $f_s$ from (1). According to Mašín (2012c), this factor may be quantified by comparing the isotropic unloading formulation of the hypoplastic model with the isotropic unloading line of the pre-defined form

\begin{equation}
\frac{\dot{\varepsilon}}{1+\varepsilon} = -K^* \frac{\dot{p}}{p} 
\end{equation}

The isotropic version of the model is obtained by algebraic manipulations with (1) (for formulation of all model components see Appendix A)

\begin{equation}
\dot{p} = \left( \frac{p}{\lambda^*} - 2f_s \frac{A_m}{g} \right) \frac{\dot{\varepsilon}}{1+\varepsilon}
\end{equation}

where

\begin{equation}
A_m = \nu_{pp}^2 \left( \frac{4\alpha_E}{\alpha_v} - 2\alpha_E^2 + 2 \frac{\alpha_E^2}{\alpha_v} - 1 \right) + \nu_{pp} \left( \frac{4\alpha_E}{\alpha_v} + 2\alpha_E \right) + 2\alpha_E + 1
\end{equation}

Comparison of (13) with (12) then yields

\begin{equation}
f_s = -\frac{3}{2A_m} \left( \frac{1}{\lambda^*} + \frac{1}{K^*} \right)
\end{equation}

The proposed model reduces to the reference one\(^2\) for $\alpha_G = \alpha_E = \alpha_v = 1$. To predict very small strain stiffness and recent stress history effects, the model must be combined with the intergranular strain concept by Niemunis and Herle (1997) (see Appendix B for its formulation). The very small strain stiffness matrix $\mathbf{M}_0$ then reads

$$
\mathbf{M}_0 = m_R f_s \mathbf{L}
$$

\(^2\) Note that additional modification of an exponent appearing in the formulation of $f_s$ is proposed, which is detailed in Appendix A.
The shear $G_{tp0}$ component of the tensor $\mathbf{M}_0$ is given by (from (2), (14) and (15))

$$G_{tp0} = m_R \frac{3p}{2A_m} \left( \frac{1}{\lambda'} + \frac{1}{\kappa'} \right) \frac{\alpha_E}{2\alpha_G} \left( 1 - \nu_{pp} - 2\frac{\alpha_E}{\alpha_G} \nu^2_{pp} \right)$$

In the present work, we consider the following dependency of $G_{tp0}$ on mean stress $p$ (Wroth and Houlsby 1985)

$$G_{tp0} = p_r A_g \left( \frac{p}{p_r} \right)^{n_g}$$

where $A_g$ and $n_g$ are parameters and $p_r$ is a reference pressure of 1 kPa. Comparison of (17) and (18) yields the following expression\(^3\) for the variable $m_R$.

$$m_R = p_r A_g \left( \frac{p}{p_r} \right)^{n_g} \frac{4A_m \alpha_G}{2p\alpha_G} \left( \frac{\lambda'\kappa'}{\lambda' + \kappa'} \right) \frac{1}{\left( 1 - \nu_{pp} - 2\frac{\alpha_E}{\alpha_G} \nu^2_{pp} \right)}$$

Complete model formulation is given in Appendices. Its finite element implementation is freely available on the web Gudehus et al. (2007).

**Model calibration**

In this section, we focus on calibration of material constants new with respect to the original model. For calibration of the parameters parameters $\phi_c$, $N$, $\lambda'$ and $\kappa'$, the readers are referred to Mašín (2012b). As discussed by Mašín and Rott (2013), complete calibration of transversely isotropic elastic models requires five measurements of wave propagation velocities:

- $V_{S0}(0^\circ)$: S-wave velocity propagating in the direction normal to the plane of isotropy.
- $V_{S0}(90^\circ)$: S-wave velocity propagating within the plane of isotropy with in-plane polarisation.
- $V_P(0^\circ)$: P-wave velocity propagating in the direction normal to the plane of isotropy.
- $V_P(90^\circ)$: P-wave velocity propagating within the plane of isotropy.
- $V_p(45^\circ)$: P-wave velocity under inclination of $45^\circ$ with respect to the plane of isotropy.

The material constants may then be calculated using

\[ G_{tp0} = E \]
\[ \alpha_G = \frac{C-D}{2E} \]
\[ \nu_{pp} = \frac{AC-B^2}{C+D} \]
\[ \alpha_v = \frac{B}{C+D} \nu_{pp} \]
\[ \alpha_E = \frac{B\alpha_G^2 - C\alpha_v \nu_{pp}(1+\nu_{pp})}{D\nu_{pp}} \]

\(^3\) Note that in the original intergranular strain concept formulation, $m_R$ is considered as a parameter. Contrary, in the formulation proposed here, $m_R$ is a variable calculated on the basis of $G_{tp0}$ expression (18).
where (from Mavko et al. 2009)

\[
C = \rho_t V_p^2(90^\circ)
\]

\[
D = C - 2\rho_t V_S^2(90^\circ)
\]

\[
A = \rho_t V_P^2(0^\circ)
\]

\[
E = \rho_t V_S^2(0^\circ)
\]

\[
B = -E + \sqrt{4\rho_t^2 V_p^2(45^\circ) - 2\rho_t V_P^2(45^\circ)(C + A + 2E) + (C + E)(A + E)}
\]

with \( \rho_t \) being soil density.

The above mentioned experiments are not routinely performed in geotechnical engineering laboratories. A simpler calibration procedure assumes that at least bender element shear velocity measurements on vertically and horizontally oriented samples are available for calibration of \( \alpha_G \). \( \alpha_E \) and \( \alpha_v \) may then be evaluated using empirical correlations proposed by Mašín and Rott (2013). Let us define the exponents \( \chi_{GE} \) and \( \chi_{Gv} \) as

\[
\alpha_E = \alpha_G^{(1/\chi_{GE})}
\]

\[
\alpha_v = \alpha_G^{(1/\chi_{Gv})}
\]

Based on evaluation of an extensive experimental database, Mašín and Rott (2013) suggested \( \chi_{GE} = 0.8 \) and \( \chi_{Gv} = 1 \). The remaining parameter \( v_{pp} \) may in this simplified calibration procedure be estimated by trial-and-error using large strain shear stiffness measurements.

**Evaluation of the model**

The proposed model has been evaluated using extensive experimental data set on London clay from Imperial College project by Nishimura et al. (2007), Nishimura (2005), Gasparre et al. (2007) and Gasparre (2005). They tested undisturbed samples of London clay from the excavation at Heathrow, Terminal 5. For the material description and details of the experimental procedures the readers are referred to the above cited publications. The parameters \( \alpha_G \), \( A_g \) and \( n_g \) were calibrated using resonant column appauratus tests on London clay. Empirical expressions were adopted for \( \alpha_E \) (30) and \( \alpha_v \) (31). \( v_{pp} \) was estimated using stress-strain curves of shear tests at large strains. The parameters \( \varphi_c, \chi^* \) and \( \kappa^* \), calibrated using data by Gasparre (2005), were taken over from Mašín (2009). The parameter \( N \) was adjusted so that the soil overconsolidation manifested by the undrained stress paths was predicted properly. The initial value of void ratio \( e = 0.69 \) was calculated from the specimen water content and specific gravity provided by Nishimura et al. (2007). The material parameters adopted in all the simulations are in Table 1 and 2. Predictions by the proposed model have been compared with predictions by the reference model by Mašín (2005).

In the evaluation, we used hollow cylinder tests on London clay from the depth of 10.5m by Nishimura (2005) and Nishimura et al. (2007). Two sets of experiments have been simulated.

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4 Note that the classical Graham and Houlsby (1983) model assumes \( \chi_{GE} = 0.5 \) and \( \chi_{Gv} = 1 \)
In the first one, the soil was isotropically consolidated to the \textit{in-situ} effective stress of \( p = 323 \text{kPa} \) (series "IC" by Nishimura et al. 2007). In the second one, the initial conditions represented the estimated anisotropic \textit{in-situ} stress state of \( p = 323 \text{ kPa} \) and \( p = -166 \text{ kPa} \) (series "AC" by Nishimura et al. 2007). In both cases, the soil was sheared after consolidation under undrained conditions with controlled vertical strain. Total stress path was defined by constant total mean stress and constant values of variables \( \alpha_{\delta\sigma} \) and \( b \). These were defined as

\[
\alpha_{\delta\sigma} = \frac{1}{2} \tan \left( \frac{2 \Delta \tau_{\theta\theta}}{\Delta \sigma_{\theta\theta} - \Delta \sigma_{\theta\theta}} \right) \tag{32}
\]

\[
b = \frac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_3} \tag{33}
\]

where \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) are the major, intermediate and minor principal stresses respectively and \( \sigma_z, \sigma_\theta \) and \( \tau_{\theta\theta} \) are rectilinear stress components in the specimen frame of reference (see Nishimura et al. 2007). The value of \( b \) represents the contribution of the intermediate principal stress such that in the standard compression experiment in triaxial apparatus \( b = 0 \). Only simulations with \( b = 0.5 \) are presented here for brevity. \( \alpha_{\delta\sigma} \) represents the principal stress inclination revealing soil anisotropy. In the standard triaxial test, \( \alpha_{\delta\sigma} = 0^\circ \) for the vertically trimmed specimen and \( \alpha_{\delta\sigma} = 90^\circ \) for the horizontally trimmed specimen.

Figure 1a demonstrates calibration of the parameter \( \alpha_G \) and predictions of very small strain stiffness anisotropy. Figure 1b shows secant stiffness \( G_{\theta\theta} \) degradation with shear strain \( \gamma_{\theta\theta} \) in hollow cylinder test with \( \alpha_{\delta\sigma} = 23^\circ \) and \( b = 0.5 \) and as predicted by the model.

Stress paths of various tests are in the \( p \) vs. \( (\sigma_2 - \sigma_\theta)/2 \) stress space plotted in Fig. 2. Stress-strain curves \( (q/p) \) vs. the principal strain difference \( \epsilon_1 - \epsilon_3 \) are presented in Fig. 3. The soil anisotropy is revealed by the deviation of the stress path from vertical (constant \( p \)). The proposed model predicts the stress path inclination properly for both isotropically and anisotropically consolidated specimens. The stress paths deviate from the experimental after the peak of \( q/p \), but this may be explained by the specimen rupture and strain localisation into shear bands (see Nishimura et al. 2007 for indication of the pre-rupture stress path portions). Predictions by the model by Mašín (2005) are shown in Figs. 2c and 3e,f for comparison. This model predicts some degree of stress-induced anisotropy in the anisotropically consolidated specimens, but its degree cannot be controlled by a parameter and in the present case it is clearly underestimated. The response of the isotropically consolidated specimens is incorrectly predicted as initially isotropic by the Mašín (2005) model.

**Summary and conclusions**

A new version of clay hypoplasticity model is developed for predicting stiffness anisotropy. The model is based on the reference model by Mašín (2012b), in which the stiffness tensor \( \mathbf{L} \) is replaced by an anisotropic elasticity tensor. The model has been evaluated using comprehensive data set on London clay, which includes measurements of the influence of anisotropy in the hollow cylinder apparatus. It is demonstrated that the proposed model predicts not only the influence of anisotropy on the very small strain stiffness, but it also improves predictions of undrained stress paths.
Acknowledgment

Financial support by the research grant P105/12/1705 of the Czech Science Foundation is greatly appreciated.

Notation and conventions

Compression negative sign convention is adopted throughout.

Tensorial operations:

\[ ||X|| \quad \text{Euclidean norm} \quad \sqrt{X_{ij}X_{ij}} \]
\[ \text{tr}X \quad \text{trace operator} \quad 1_{ij}X_{ij} \]
\[ \mathcal{L} : Y \quad \text{inner product} \quad L_{ijkl}Y_{kl} \]
\[ X \otimes Y \quad \text{outer product} \quad X_{ij}Y_{kl} \]
\[ X \cdot Y \quad \text{inner product} \quad X_{ij}Y_{jk} \]
\[ X \circ Y \quad \text{tensor product} \quad \frac{1}{2}(X_{ik}Y_{jl} + X_{il}Y_{jk} + X_{jl}Y_{ik} + X_{jk}Y_{il}) \]
\[ \dot{X} \quad \text{rate of} \quad X \]
\[ \dot{\dot{X}} \quad \text{objective (Jaumann) rate of} \quad X \]
\[ \ddot{X} \quad \text{tensor normalised by its Euclidean norm} \quad \ddot{X} = X/||X|| \]

Variables:

1. second-order identity tensor
2. Cauchy effective stress tensor
p. mean effective stress
\( \dot{\varepsilon} \). Euler stretching tensor
e. void ratio
\( \mathcal{L} \). hypoplasticity fourth-order tensor
N. hypoplasticity second-order tensor
\( f_s \). barotropy factor of hypoplastic equation
\( f_d \). pyknotropy factor of hypoplastic equation
\( a_1, a_2, a_3, a_4, a_5 \). parameters of transversely isotropic elasticity model
A, B, C, D, E. parameters of transversely isotropic elasticity model
n. unit vector normal to the plane of symmetry
p. second order tensor \( p_{ij} = n_i n_j \)
\( G_{tp}, G_{pp} \). shear moduli ("p" in-plane direction, "t" transversal direction)
\( E_t, E_p \). Young moduli ("p" in-plane direction, "t" transversal direction)
\( v_{tp}, v_{pp} \). Poisson ratios ("p" in-plane direction, "t" transversal direction)
\( \alpha_G \). anisotropy ratio of shear moduli
\( \alpha_E \) anisotropy ratio of Young moduli
\( \alpha_v \) anisotropy ratio of Poisson ratios
\( \chi_{GE}, \chi_{GV} \) exponents of transversely elastic model formulation
\( N \) hypoplastic model parameter (position of normal compression line)
\( \lambda^* \) hypoplastic model parameter (slope of normal compression line)
\( \kappa^* \) hypoplastic model parameter controlling volumetric unloading response
\( \varphi_c \) critical state friction angle
\( A_m \) variable in hypoplastic model formulation
\( \mathcal{M}_0 \) very small strain stiffness tensor
\( m_r \) variable controlling very small strain shear modulus
\( G_{tp0} \) shear modulus at very small strain
\( p_r \) reference stress equal to 1 kPa
\( A_g \) parameter quantifying the dependency of \( G_{tp0} \) on mean stress
\( n_g \) parameter quantifying the dependency of \( G_{tp0} \) on mean stress
\( V_{SH}(X^o) \) S-wave velocity (\( X \) is the direction of propagation with respect to the axis of symmetry)
\( V_P(X^o) \) P-wave velocity (\( X \) is the direction of propagation with respect to the axis of symmetry)
\( \rho_t \) soil density
\( \alpha_{\sigma\sigma} \) principal stress increment inclination in hollow cylinder test
\( \Delta X \) Finite increment of \( X \)
\( \sigma_1, \sigma_2, \sigma_3 \) principal stresses (major, intermediate, minor)
\( \epsilon_1, \epsilon_3 \) principal strains (major, minor)
\( \sigma_2, \sigma_\theta, \tau_{z\theta} \) rectilinear stress components in the specimen frame of reference in hollow cylinder apparatus
\( b \) variable quantifying the intermediate principal stress magnitude in hollow cylinder apparatus

**Hypoplasticity specific variables (appearing in appendices only):**

\( f^A_d \) pyknotropy factor of hypoplastic equation, asymptotic state value
\( \mathcal{A} \) fourth-order tensor in hypoplastic model formulation
\( p_e \) Hvorslev equivalent pressure
\( d^A \) second-order tensor specifying asymptotic strain rate direction
\( d \) normalised second-order tensor specifying asymptotic strain rate direction
\( \alpha_f \) hypoplastic variable controlling rate of stiffness decrease, may be considered as a model parameter
\( F_m \) factor of Matsuoka-Nakai yield condition
\( \omega \) variable controlling asymptotic state boundary surface shape
\( a \) hypoplastic variable controlling peak strength, which may be considered as a model parameter
\( I_1, I_2, I_3 \) stress invariants
\( \cos 3\theta \) Lode angle function
\( \xi \) variable controlling asymptotic strain rate direction
\( a_f \) variable in hypoplastic model formulation
\( R, m_r, \beta_r, \chi \) intergranular strain concept parameters
\( \mathbf{M} \) stiffness tensor of the intergranular strain concept formulation

\( m_T \) variable in the intergranular strain concept formulation

\( \mathbf{I} \) fourth-order identity tensor

\( \mathbf{\delta} \) intergranular strain tensor

\( \rho \) normalised intergranular strain tensor magnitude

\( \mathbf{0} \) second-order null tensor

**Appendix A**

The Appendix summaries remaining equations of the proposed hypoplastic model which have not been specified in the main text.

\[
\dot{\mathbf{\sigma}} = f_3 \mathbf{L} : \dot{\mathbf{\varepsilon}} - \frac{f_d}{f_d^*} \mathbf{A} : \mathbf{d} \| \dot{\mathbf{\varepsilon}} \| \tag{34}
\]

\[
\mathbf{A} = f_3 \mathbf{L} + \frac{\sigma}{\lambda_*} \otimes \mathbf{1} \tag{35}
\]

\[
f_d = \left( \frac{2p}{p_e} \right)^{\alpha_f} \tag{36}
\]

\[
p_e = p_f \exp \left[ \frac{N(1+e)}{\lambda_*} \right] \tag{37}
\]

\[
f_d^A = 2^{\alpha_f} (1 - F_m)^{\alpha_f/\omega} \tag{38}
\]

\[
F_m = \frac{g_{_3} + h_{_1} l_{_2}}{l_{_3} + h_{_1} l_{_2}} \tag{39}
\]

\[
\omega = -\frac{\ln(\cos^2 \varphi_c)}{\ln 2} + a(F_m - \sin^2 \varphi_c) \tag{40}
\]

\[
I_1 = \text{tr} \; \mathbf{\sigma} \tag{41}
\]

\[
I_2 = \frac{1}{2} [\mathbf{\sigma} : \mathbf{\sigma} - (I_1)^2] \tag{42}
\]

\[
I_3 = \det \mathbf{\sigma} \tag{43}
\]

\[
\mathbf{d} = \frac{\mathbf{d}^A}{\| \mathbf{d}^A \|} \tag{44}
\]

\[
\mathbf{d}^A = -\dot{\mathbf{\theta}}^* + \mathbf{1} \left[ \frac{2}{3} - \frac{\cos 3\theta + 1}{4} (F_m)^{1/4} \right] \frac{(F_m)^{\xi/2} - \sin \xi \varphi_c}{1 - \sin \xi \varphi_c} \tag{45}
\]

\[
\cos 3\theta = -\sqrt{6} \frac{\text{tr}(\dot{\mathbf{\theta}}^* \cdot \dot{\mathbf{\theta}}^*)}{[\mathbf{\theta}^* \cdot \mathbf{\theta}^*]^{3/2}} \tag{46}
\]

\[
\xi = 1.7 + 3.9 \sin^2 \varphi_c \tag{47}
\]
\[ \sigma^* = \frac{\sigma}{\text{tr} \sigma} - \frac{1}{3} \quad (48) \]

The exponent \( \alpha_f \) controls irreversibility of the deformation inside the asymptotic state boundary surface. In fact, for high values of \( \alpha_f \) the response of the basic model is practically reversible inside the asymptotic state boundary surface with \( f_s \mathbf{L} \) being the stiffness matrix. The model predictions then resemble predictions by the critical state elasto-plastic models. In the reference model by Mašín (2012b), a fixed value of \( \alpha_f = 2 \) has been suggested. Thorough evaluation of the model non-linear properties, however, indicated that \( \alpha_f \) value by the Mašín (2005) model leads to better predictions. It is thus suggested to use

\[
\alpha_f = \frac{\ln \left( \frac{\lambda^* + \kappa^* (\frac{3 + a_f^2}{a_f^2})}{\lambda^* + \kappa^* (\frac{3 + \kappa^2}{\kappa^2})} \right)}{\ln 2} \quad (49)
\]

\[
\alpha_f = \frac{\sqrt{3} (3 - \sin \varphi_C)}{2 \sqrt{2} \sin \varphi_C} \quad (50)
\]

In fact, if needed, \( \alpha_f \) can be considered as a model parameter controlling non-linear response inside the asymptotic state boundary surface in the case \( v_{pp} \) is calibrated rigorously using wave velocity measurements. The model assumes parameters \( \varphi_C, \lambda^*, \kappa^*, N \) and \( v_{pp} \) and state variable \( e \) (void ratio). \( \alpha \) is controlling peak friction angle, standard value of \( \alpha = 0.3 \) was suggested by Mašín (2012b). If required, the value of \( \alpha \) can be modified to control peak friction angle, see Mašín (2012b) for details. \( \alpha_G, \alpha_E \) and \( \alpha_v \) are parameters controlling stiffness anisotropy; \( \alpha_E \) and \( \alpha_v \) may be approximated using empirical formulations.

**Appendix B**

In this appendix, we summarise the version of the intergranular strain concept used in the proposed model. The intergranular strain concept was originally proposed by Niemunis and Herle (1997).

\[
\delta = \mathbf{M} : \dot{\epsilon} \quad (51)
\]

\[
\mathbf{M} = [\rho^x m_T + (1 - \rho^x)m_R] f_s \mathbf{L} + \begin{cases} 
\rho^x (1 - m_T) f_s \mathbf{L} : \delta \otimes \delta + \rho^x f_s f_d \mathbf{N} \delta & \text{for } \delta : \dot{\epsilon} > 0 \\
\rho^x (m_R - m_T) f_s \mathbf{L} : \delta \otimes \delta & \text{for } \delta : \dot{\epsilon} \leq 0 
\end{cases} \quad (52)
\]

\[
\rho = \frac{\| \delta \|}{R} \quad (53)
\]

\[
\delta = \begin{cases} 
\delta / \| \delta \| & \text{for } \delta \neq 0 \\
0 & \text{for } \delta = 0
\end{cases} \quad (54)
\]

\[
\delta = \begin{cases} 
(I - \dot{\delta} \otimes \dot{\delta} \rho^{\zeta_r}) : \dot{\epsilon} & \text{for } \delta : \dot{\epsilon} > 0 \\
\dot{\epsilon} & \text{for } \delta : \dot{\epsilon} \leq 0
\end{cases} \quad (55)
\]
\[ m_R = p_r A_g \left( \frac{p}{p_r} \right)^{n_g} \frac{4A_m a_G}{2p a_G} \left( \frac{\lambda^*}{\lambda^* + \kappa^*} \right) \frac{1}{(1 - \nu_p - 2a_G^2/\alpha_{pp}^2)} \]  
\[ m_T = m_{rat} m_R \]

\( A_g, \ n_g, \ m_{rat}, \ \beta_r, \ \chi \) are parameters and \( \delta \) is state variable.

References


**Tables**

Table 1: Parameters of the intergranular strain concept by Niemunis and Herle (1997) adopted in combination with different hypoplastic models.

<table>
<thead>
<tr>
<th>model</th>
<th>( A_g ) or ( m_R )</th>
<th>( n_g )</th>
<th>( m_{rat} ) or ( m_T )</th>
<th>( R )</th>
<th>( \beta_T )</th>
<th>( \chi )</th>
</tr>
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<tbody>
<tr>
<td>proposed model</td>
<td>270</td>
<td>1</td>
<td>0.5</td>
<td>5 \times 10^{-5}</td>
<td>0.08</td>
<td>0.9</td>
</tr>
<tr>
<td>Mašín (2005) model</td>
<td>8</td>
<td>n/a</td>
<td>4</td>
<td>5 \times 10^{-5}</td>
<td>0.08</td>
<td>0.9</td>
</tr>
</tbody>
</table>

Table 2: Parameters of the hypoplastic models used in simulations.

<table>
<thead>
<tr>
<th>model</th>
<th>( \varphi_c )</th>
<th>( \lambda^* )</th>
<th>( \kappa^* )</th>
<th>( N )</th>
<th>( \nu_{pp} ) or ( r )</th>
<th>( \alpha_G )</th>
</tr>
</thead>
<tbody>
<tr>
<td>proposed model</td>
<td>21.9(^\circ)</td>
<td>0.095</td>
<td>0.015</td>
<td>1.19</td>
<td>( \nu_{pp} = 0.1 )</td>
<td>2</td>
</tr>
<tr>
<td>Mašín (2005) model</td>
<td>21.9(^\circ)</td>
<td>0.095</td>
<td>0.015</td>
<td>1.19</td>
<td>( r = 0.3 )</td>
<td>n/a</td>
</tr>
</tbody>
</table>

**Figure captions**

Figure 1: (a) \( \alpha_G \) calibration based on experiments by Nishimura (2005) and Gasparre (2005). (b) Secant shear stiffness degradation as measured by Nishimura (2005) in hollow cylinder test with \( \alpha_{d\sigma} = 23^\circ \) and \( b = 0.5 \) and predictions by the proposed model.

Figure 2: Stress paths in the \( p \) vs. (\( \sigma_2 - \sigma_3 \))/2 stress space: Experimental data by Nishimura et al. (2007), proposed model and Mašín (2005) model predictions.

Figure 3: The ratio \( q/p \) vs. the principal strain difference \( \varepsilon_1 - \varepsilon_3 \) for three simulation sets: Experimental data by Nishimura et al. (2007), proposed model and Mašín (2005) model predictions.
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(b) Secant shear stiffness degradation as measured by Nishimura (2005) in hollow cylinder test with $\alpha_{do} = 23^\circ$ and $b = 0.5$ and predictions by the proposed model.
Figure 2: Stress paths in the $p'$ vs. $(\sigma_z - \sigma_\theta)/2$ stress space: Experimental data by Nishimura et al. (2007), proposed model and Mašín (2005) model predictions.
Figure 3: The ratio $q/p'$ vs. the principal strain difference $\epsilon_1 - \epsilon_3$ for three simulation sets: Experimental data by Nishimura et al. (2007), proposed model and Mašín (2005) model predictions.