Comparison of predictive capabilities of selected elasto-plastic and hypoplastic models for structured clays

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Abstract

Different approaches to constitutive modelling of natural structured clays are in the paper compared by means of experimental data on natural Pisa and Bothkennar clays. The models evaluated are a hypoplastic model for structured clays, its simple elasto-plastic equivalent that requires parameters with similar physical meaning, and advanced elasto-plastic models based on kinematic hardening approach. Hypoplasticity predicts non-linear stress-strain response in the pre-failure region and different stiffness in different loading directions, it thus provides a clear qualitative advance with respect to the simple elasto-plastic model. It gives qualitatively similar predictions with the kinematic hardening models. The structure degradation and the large-strain response are predicted similarly by both the hypoplastic and elasto-plastic models, which shows that the critical state soil mechanics theories can be treated successfully within the framework of the theory of hypoplasticity.

Keywords: Constitutive relations; hypoplasticity; elasto-plasticity; clays; structure of soils

International Geotechnical Classification Numbers: E13

Introduction

In recent years, many constitutive models for structured clays have been developed. They usually share the conceptual approach for the incorporation of soil structure, which is based on description of the behaviour of appropriate reference (sometimes denoted as ”destructured”) material, and addition of structure through one or more additional state variables that characterises the degree of bonding between soil particles and/or state of soil fabric (see, e.g., Lagioia and Nova (1995) and Cotecchia and Chandler (2000)). This approach is advantageous as the models may be developed ”hierarchically” (Muir Wood and Gajo, 2005) by including the soil structure into existing constitutive models for reference material.

Naturally, the models for structured soils share merits and shortcomings with their reference counterparts. Therefore, their predictive capabilities may differ quite significantly, although they use the same concepts for the incorporation of structure. In this paper, predictions by several constitutive models for structured clays of different complexities are compared with respect to two sets of experimental data on natural clays. The direct comparison of different models should help the potential users in choosing suitable model for solving problems they are confronted to.

The aim of the paper is to demonstrate merits of a less common approach to constitutive modelling of geomaterials, hypoplasticity. Predictions by a recently proposed hypoplastic model for structured clays by Mašín (2007) are compared with predictions by elasto-plastic models from two groups. First, a simple elasto-plastic counterpart of the hypoplastic model is developed. This model requires the same number of soil parameters with equivalent physical interpretation as the hypoplastic model of interest. Second, predictions by hypoplasticity are compared with predictions by advanced elasto-plastic models based on kinematic hardening approach (Baudet and Stallebrass (2004) and Rouainia and Muir Wood (2000)). These models require larger number of material constants, but they provide more realistic predictions of non-linear soil behaviour than the basic elasto-plastic critical state models. Predictions by the kinematic hardening models used for demonstration of their capabilities in this paper have been performed and published by their developers themselves.

A hypoplastic model for clays with meta-stable structure

A hypoplastic model for clays with meta-stable structure (Mašín, 2007) has been developed by modifying the basic hypoplastic model for clays by Mašín (2005). The rate formulation of hypoplastic models under consideration is characterised by a single equation (Gudehus, 1996)[1]

$$\dot{\boldsymbol{\sigma}} = f_s \boldsymbol{\mathcal{L}} : \dot{\boldsymbol{\epsilon}} + f_s f_d \mathbf{N} \left\| \dot{\boldsymbol{\epsilon}} \right\| \tag{1}$$

where $\boldsymbol{\mathcal{L}}$ and $\mathbf{N}$ are fourth- and second-order constitutive tensors respectively, f_s and f_d are two scalar factors, symbol ':' between two tensors denotes inner product with double contraction and $\|\dot{\boldsymbol{\epsilon}}\| = \sqrt{\dot{\boldsymbol{\epsilon}} : \dot{\boldsymbol{\epsilon}}}$ denote Euclidean norm of $\dot{\boldsymbol{\epsilon}}$. Cauchy stress $\boldsymbol{\sigma}$ and void ratio e are considered as state variables. The Eq. (1) is non-linear in $\dot{\boldsymbol{\epsilon}}$ and, unlike in elasto-plasticity, there is no need for splitting the strain rate into elastic and plastic parts and for introducing switch function to distinguish between elastic loading and elasto-plastic unloading.

Still, the hypoplastic models are capable of predicting the basic features of soil behaviour (see Gudehus and Mašín (2008)), such as different stiffness upon loading and unloading and, in general, in different stretching directions (Mašín et al., 2006), the influence of overconsolidation ratio on stiffness and peak strength (Hájek and Mašín, 2006) and critical state and state boundary surface (SBS) (Mašín and Herle, 2005a). The basic hypoplastic model for clays requires only five parameters. The first one, φ_c, is the critical state friction angle. Parameters N and λ^* define the position and shape of the isotropic virgin compression line with the formulation according to Butterfield (1979):

$$\ln(1+e) = N - \lambda^* \ln\left(\frac{p}{p_r}\right) \tag{2}$$

where p_r is the reference stress 1kPa. The parameter κ^* determines bulk modulus at overconsolidated states and the parameter r controls shear modulus. The parameters have therefore similar physical interpretation as parameters M, N, λ, κ and G of the Modified Cam clay model by Roscoe and Burland (1968).

The basic hypoplastic model has been modified for clays with meta-stable structure by introducing additional state variable sensitivity s that measures the degree of soil structure and by incorporating a suitable structure degradation law (Mašín, 2007, 2006). Sensitivity is defined as the ratio of the sizes of SBS of structured and reference materials. It is measured along a constant volume section through SBS, see Fig. 1. The rate formulation of sensitivity reads

$$\dot{s} = -\frac{k}{\lambda^*}(s - s_f)\dot{\epsilon}^d \tag{3}$$

where k, s_f and λ^* are parameters and $\dot{\epsilon}^d$ is the damage strain rate, defined as

$$\dot{\epsilon}^d = \sqrt{(\dot{\epsilon}_v)^2 + \frac{A}{1-A}(\dot{\epsilon}_s)^2} \tag{4}$$

$\dot{\epsilon}_v$ and $\dot{\epsilon}_s$ denote volumetric and shear strain rates respectively and A is a model parameter. The

[1] To be more precise, the rate formulation of hypoplastic models reads $\mathring{\boldsymbol{\sigma}} = f_s \boldsymbol{\mathcal{L}} : \mathbf{D} + f_s f_d \mathbf{N} \|\mathbf{D}\|$, where $\mathring{\boldsymbol{\sigma}}$ is the objective stress rate and $\mathbf{D}$ the Euler's stretching tensor.

parameter k controlls the rate of structure degradation, s_f is the final sensitivity and A controls the relative influence of volumetric and shear strain rates on structure degradation. The complete mathematical formulation of the hypoplastic model for structured clays is given in Appendix A. Its single- and finite-element implementation is freely available (see Gudehus et al. (2007)).

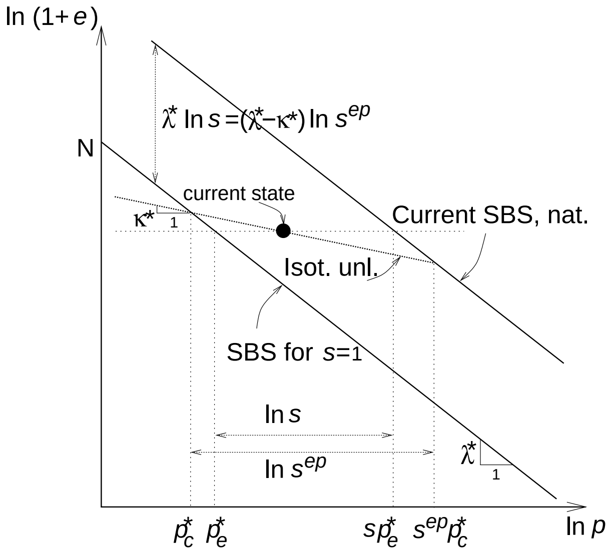


Figure 1: Definitions of sensitivities s and s^{ep}, quantities p_c^* and p_e^* and material parameters N, λ^* and κ^*.

An elasto-plastic equivalent of the hypoplastic model

In order to highlight merits of the hypoplastic formulation, predictions by the hypoplastic model are in this paper compared with its elasto-plastic "equivalent". The elasto-plastic model used requires the same number of material parameters with similar physical interpretation as the hypoplastic model - it is therefore based on the Modified Cam clay model, with Butterfield's (1979) compression law (Eq. (2)) and a structure degradation law equivalent to Eqns. (3) and (4). The model is thus conceptually similar to a number of existing single-hardening elasto-plastic models for structured soils (e.g., Liu and Carter (2002); Lagioia and Nova (1995)). The same approach to incorporate the structure into existing elasto-plastic models has already been used e.g. by Baudet and Stallebrass (2004), Rouainia and Muir Wood (2000), Kavvadas and Amorosi (2000) and Gajo and Muir Wood (2001) to enhance models based on kinematic hardening approach. Predictions of some of these models will be shown in the Evaluation section of this paper.

In the model (denoted here as "Structured Modified Cam clay model, SMCC"), sensitivity (s^{ep}) is included as an additional state variable as in hypoplasticity, but it is measured along the *elastic wall*, not along the constant volume section through SBS (see Fig. 1). s^{ep} thus represents the ratio of the sizes of yield surfaces of natural and reference materials. From Fig. 1 it is clear that

$$s^{ep} = s^{\left(\frac{\lambda^*}{\lambda^*-\kappa^*}\right)} \tag{5}$$

The rate formulation for sensitivity s^{ep} reads

$$\dot{s}^{ep} = -\frac{k}{\lambda^* - \kappa^*}(s^{ep} - s_f^{ep})\dot{\epsilon}^d \tag{6}$$

and the damage strain rate is defined as

$$\dot{\epsilon}^d = \sqrt{(\dot{\epsilon}_v^p)^2 + \frac{A}{1-A}(\dot{\epsilon}_s^p)^2} \tag{7}$$

where $\dot{\epsilon}_v^p$ and $\dot{\epsilon}_s^p$ denote *plastic* volumetric and shear strain rates respectively. A complete mathematical formulation of the SMCC model is given in Appendix B.

From Eqs. (3,4) and (6,7) it is clear that the structure degradation laws of hypoplastic and SMCC models are not exactly the same – $\dot{\epsilon}^d$ is for the SMCC model defined in terms of plastic strain rates, rather than in terms of total strain rates as in hypoplasticity, and the parameter κ^* enters the formulation of $\dot{s}^{ep}$ in order to preserve the influence of the parameter k on the rate of structure degradation.

To show differences in the two formulations of structure degradation laws, simulations of the isotropic compression test on isotropically normally consolidated specimens with varying parameter k and $s_f = 1$ are plotted in Fig. 2. The figure demonstrates that for the same values of the parameter k both laws yield for all practical purposes equivalent rates of structure degradation.

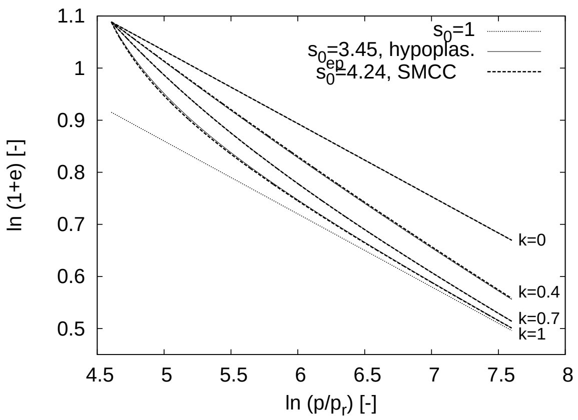


Figure 2: Demonstration of similarity of the two structure degradation laws by simulation of an isotropic compression test.

To demonstrate this issue in more detail, predictions by the two models are compared using the concept of the *normalised incremental stress response envelopes* (NIREs, see Fig. 3). They have been introduced in Mašín and Herle (2005b) and follow directly from the concept of incremental response envelopes (Tamagnini et al., 2000) and rate response envelopes (Gudehus (1979), Gudehus and Mašín (2008)).

Figure 4 shows the state boundary surfaces and NIREs predicted by the two models for Pisa clay parameters (Tab. 1) and different strain levels. The structure degradation is significantly activated in the large strain range and it also follows that the shape of NIREs for this range (Figure 4b) is similar for the two models, the differences in predictions are mostly caused by different shapes of the state boundary surfaces. On the other hand, the predictions are different in the small- to

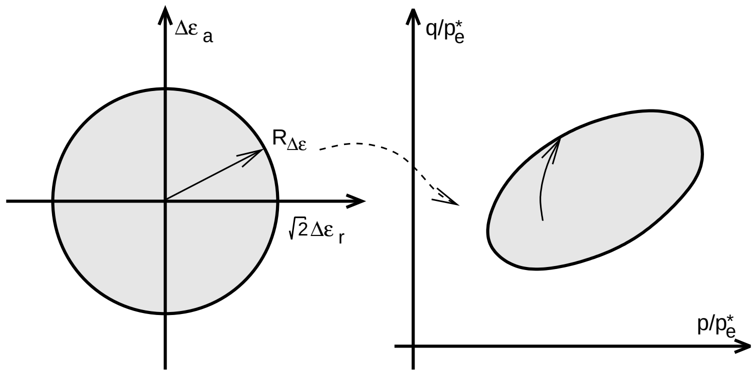


Figure 3: Demonstration of the *normalised incremental stress response envelopes* for axisymmetric conditions. p_e^* is the Hvorslev equivalent pressure.

medium strain range (before the state reaches the SBS, Fig. 4a). The elasto-plastic model predicts NIREs centred about the initial state, whereas the hypoplastic model predicts NIREs translated with respect to the initial state and thus predicts different tangent stiffness for different loading directions. It represents better the measured soil behaviour, as shown by Mašín et al. (2006) and as shown further in this paper.

From the above it can be concluded that a direct comparison of hypoplastic and SMCC models is possible and that the differences in predictions by the models are caused by different forms of the basic models, rather than by slightly different structure degradation laws.

Evaluation of the models

The models will be compared using two experimental data sets - tests on natural and reconstituted Pisa clay by Callisto (1996); Callisto and Calabresi (1998) and tests on natural Bothkennar clay by Smith et al. (1992). Predictions by the hypoplastic model from Mašín (2007) will be compared with predictions by the SMCC model and with predictions by the kinematic hardening models by Callisto et al. (2002) and Baudet (2001). Predictions by the kinematic hardening models have been performed and published by their developers themselves.

Pisa clay

Callisto and Calabresi (1998) reported laboratory experiments on natural Pisa clay. Drained probing tests were performed, with rectilinear stress paths having different orientations in the stress space. In addition to the tests on natural Pisa clay, experiments with the same stress paths were performed on reconstituted clay. Tests are labelled by prefix 'A' and 'R' for natural and reconstituted clay respectively, followed by the angle of stress paths in the $q:p$ space (measured in degrees anti-clockwise from the isotropic loading direction).

All parameters with the exception of parameters that control the influence of structure (φ_c/M, λ^*, κ^*, N and r/G) were found by simulating experiments on *reconstituted* Pisa clay. Fig. 5a shows predictions of the isotropic compression test used for calibration of the parameters N, λ^* and κ^* and Fig. 5b predictions of the shear test used for calibration of the parameters that control the

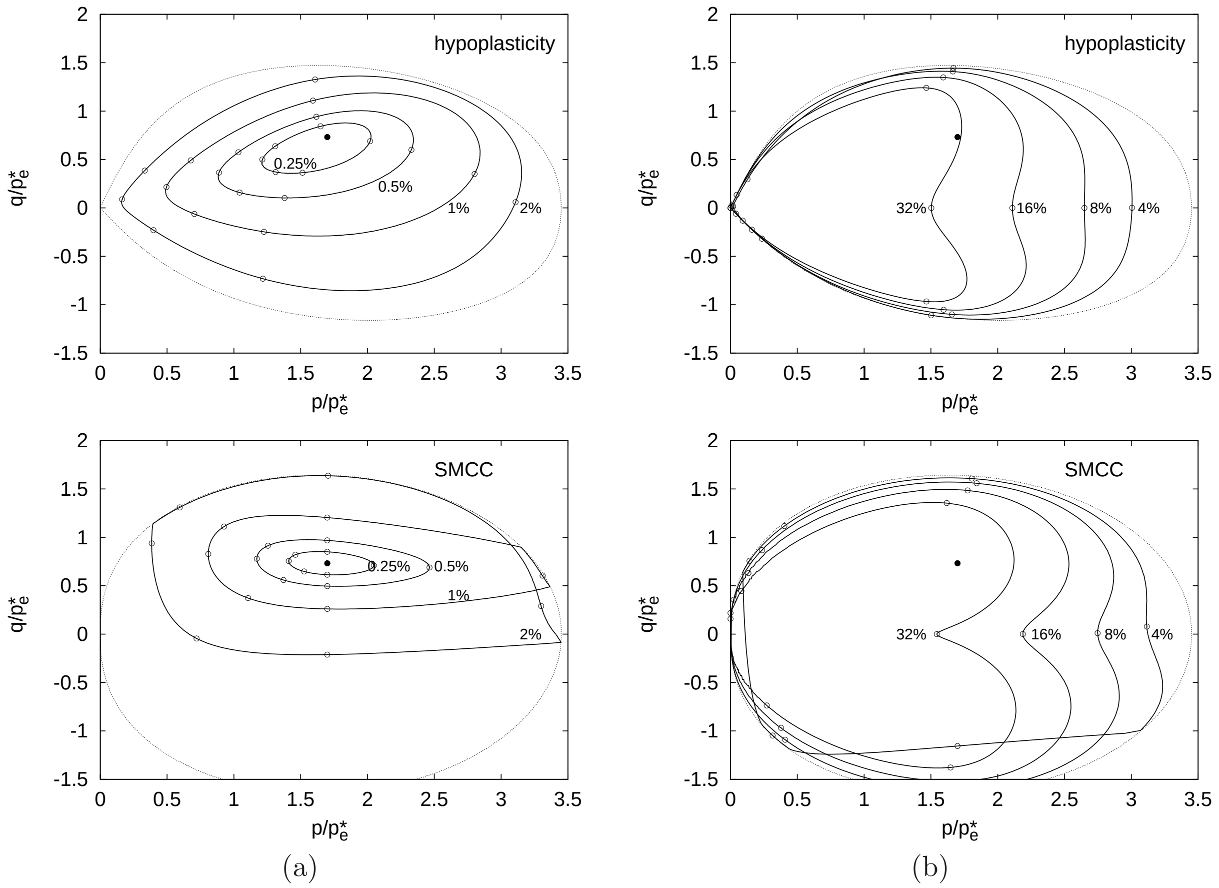
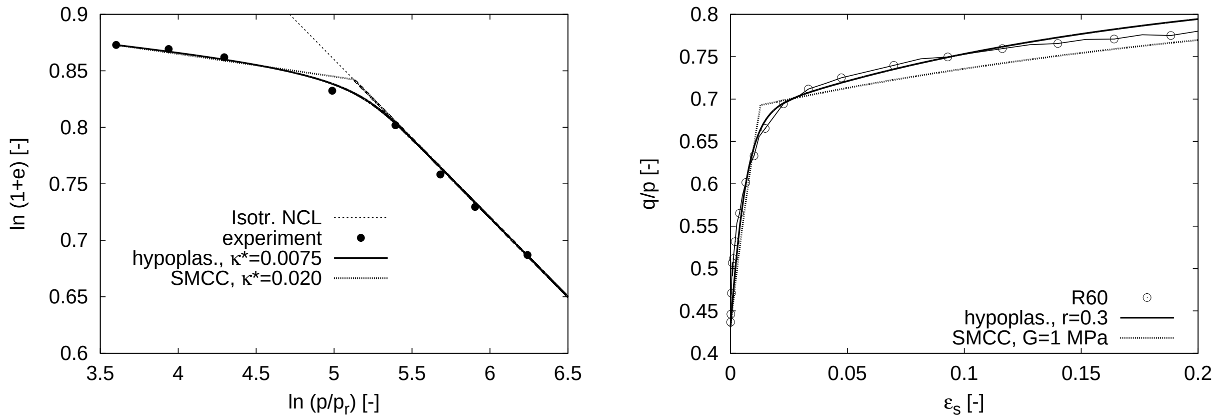


Figure 4: Normalised incremental stress response envelopes of the hypoplastic (top) and SMCC (bottom) models plotted for medium (a) and large (b) strain range ($R_{\Delta\epsilon}$ from Fig. 3 is indicated).

shear stiffness, i.e. G (SMCC) and r (hypoplasticity). Critical state friction angle φ_c has been found by evaluation of data from all shear tests available. The structure-related parameters k, A, and s_f/s_f^{ep} were calibrated by direct evaluation of experimental data on natural Pisa clay. The approach used for their calibration, described in detail by Mašín (2007), has been proposed by Callisto and Rampello (2004). It has been assmed that the experimental procedures adopted for preparation of reconstituted clay samples correctly reproduces the stress history of Pisa clay deposit. Consequently, stress paths of tests on natural and reconstituted specimens plotted in the stress space normalised with respect to volume and structure (normalised by $p_e^* s$) should coincide.This assumption allows us to find the most suitable values of A and k. Parameters of the hypoplastic model for natural Pisa clay are summarised in Table 1 and the initial values of state variables in Table 2.

The experiments on natural Pisa clay have been simulated by Callisto et al. (2002) using a modified kinematic hardening model for structured clays by Rouainia and Muir Wood (2000). When compared with the SMCC model, the kinematic hardening model takes into account non-linearity of soil behaviour inside the state boundary surface, small-strain stiffness anisotropy, non-circular cross section of the yield locus in the octahedral plane and fabric anisotropy, which is included through rotated shape of the SBS. The model is thus in a sense more evolved than the hypoplastic model, which does not consider anisotropy explicitly in its formulation. This enhancement is payed by a larger number of material constants (11 parameters of the kinematic hardening model, compared



(a) (b)

Figure 5: (a) Calibration of the parameters N, λ^* and κ^* of hypoplastic and SMCC models on the basis of an isotropic compression test on reconstituted Pisa clay (data from Callisto (1996)); (b) Calibration of the parameter r of the hypoplastic model and G of the SMCC model (data from Callisto and Calabresi (1998)).

with 7 parameters of the hypoplastic model – the kinematic hardening model assumes $s_f = 1$). For details of the model formulation, calibration using Pisa clay data and simulation of the Pisa clay experiments see Callisto et al. (2002).

Table 1: Parameters of the hypoplastic and SMCC models for natural Pisa and Bothkennar clays.

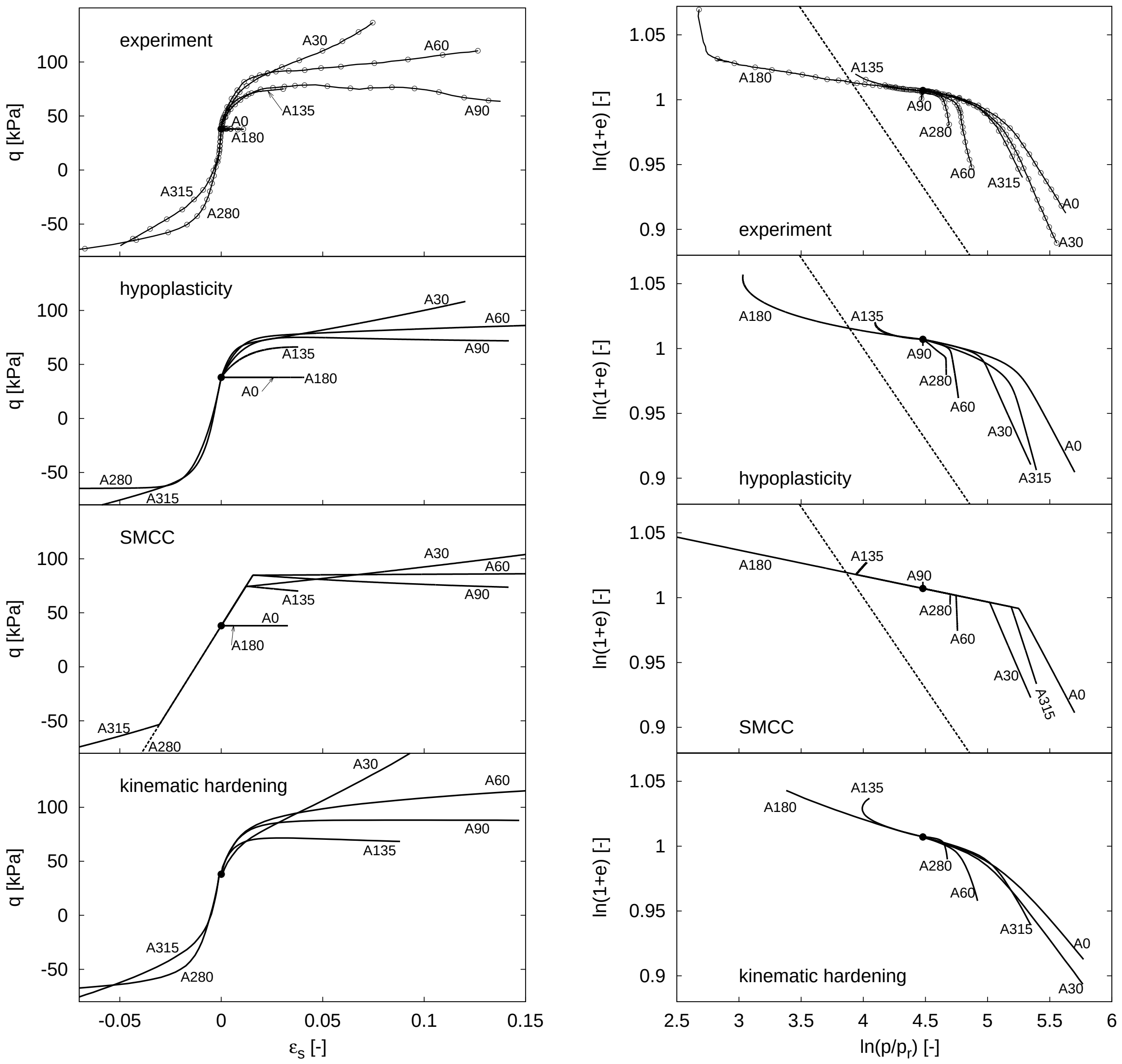
hypoplasticity	φ_c	λ^*	κ^*	N	r	k	A	s_f
Pisa clay	21.9°	0.14	0.0075	1.56	0.3	0.4	0.1	1
Bothkennar clay	35°	0.119	0.003	1.344	0.07	0.35	0.5	1
SMCC	M	λ^*	κ^*	N	G	k	A	s_f^{ep}
Pisa clay	0.85	0.14	0.02	1.56	1 MPa	0.4	0.1	1
Bothkennar clay	1.42	0.119	0.01	1.344	2 MPa	0.35	0.5	1

Table 2: *The initial values of the state variables for natural Pisa and Bothkennar clays.*

	p [kPa]	q [kPa]	e	s	s^{ep}
Pisa clay	88.2	38	1.738	3.45	4.24
Bothkennar clay	34	18	1.88	6	7.07

Fig. 6 shows the results of the simulations of experiments on Pisa clay, namely ϵ_s vs. q graphs (6a) and response in $\ln(p/p_r)$ vs. $\ln(1+e)$ plane (6b). These figures demonstrate some common features and some differences in predictions by the hypoplastic and SMCC models. Both models predict, in general, a similar stress-strain behaviour at larger strains. As already discussed in the previous paragraph, this shows that hypoplastic models may be enhanced by the structure effects in a conceptually similar way as the elasto-plastic critical state models. The main difference stems from the non-linear character of the hypoplastic equation that facilitates the non-linear response also inside the SBS, with a gradual decrease of shear and bulk moduli and a smooth structure-degradation process.

Predictions by the kinematic hardening model are qualitatively similar to the hypoplastic model. Both the models predict non-linear stress-strain response also inside the state boundary surface. Advanced features of the kinematic hardening model (incorporation of anisotropy) lead in some cases to qualitatively better predictions as compared to the hypoplastic model (e.g., ϵ_s vs. q response for tests A30 and A315).

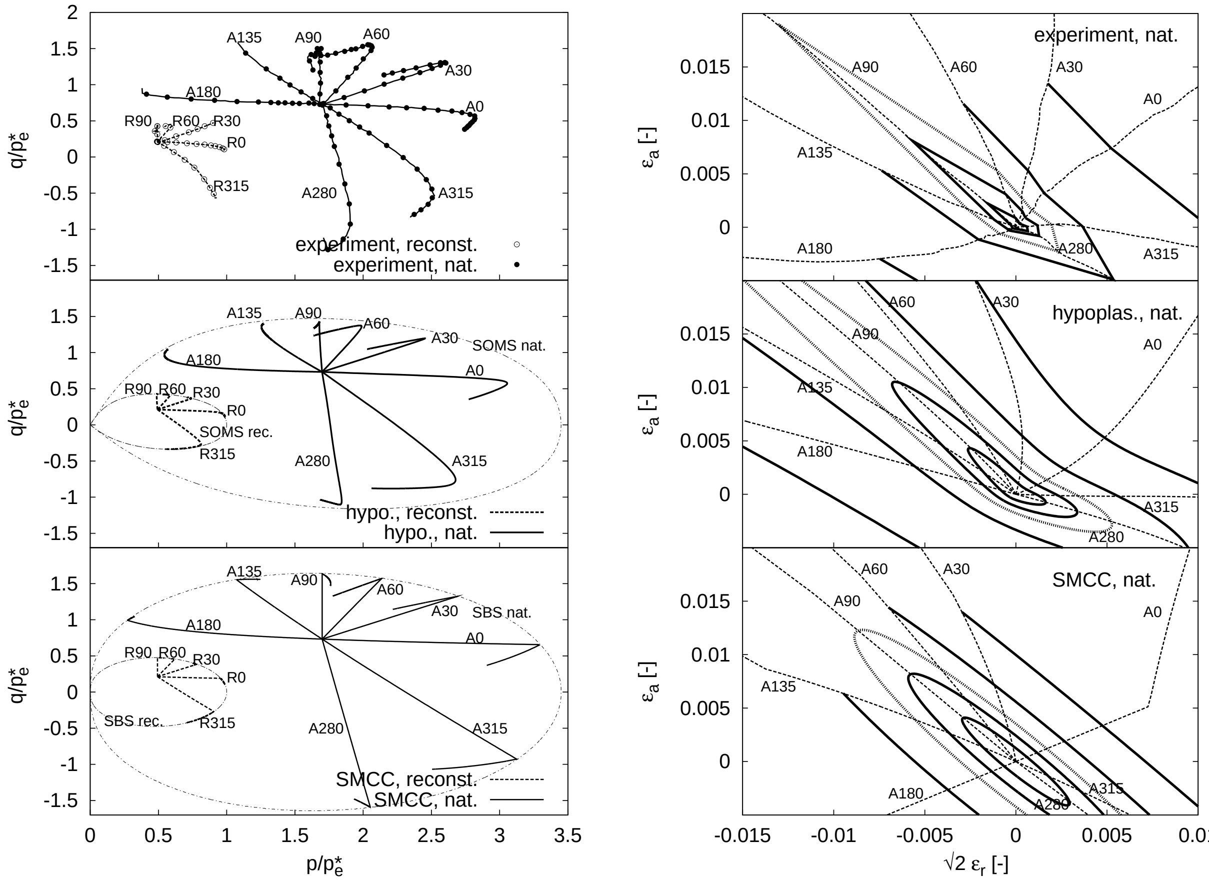


(a) (b)

Figure 6: Experiments on natural Pisa clay plotted in the ϵ_s vs. q plane (a) and $\ln(p/p_r)$ vs. $\ln(1+e)$ plane (b). Experimental data (Callisto and Calabresi, 1998), predictions by the hypoplastic, SMCC and kinematic hardening (Callisto et al., 2002) models.

Fig. 7a shows stress paths normalised by the Hvorslev equivalent pressure p_e^*, predicted by the hypoplastic and SMCC models. Both the models predict an apparently similar shape of the SBS. The hypoplastic model, however, predicts smooth structure-degradation process that takes place also inside the SBS, which reproduces better the measured data.

Performance of the models in the strain space is evaluated in Fig. 7b using the concept of incremental strain response envelopes (ISREs) (Tamagnini et al. (2000), Mašín et al. (2006)), defined inversely to the incremental stress response envelopes, which have been introduced in the previous section. The SMCC model predicts elastic behaviour inside the yield surface, i.e. elliptical response envelopes centred about the origin. The soil behaviour is clearly inelastic with softer response for compression tests, and this behaviour is reproduced correctly by the hypoplastic model.



(a) (b)

Figure 7: Normalised stress paths (a) and incremental strain response envelopes (b) of experiments on natural Pisa clay. Experimental data and predictions by the hypoplastic and SMCC models.

Bothkennar clay

Smith et al. (1992) performed a series of triaxial stress probing tests on natural Bothkennar clay. The stress-probing experiments with constant direction of stress paths in the stress space are labelled by prefix 'LCD' followed by the orientation of the stress paths in $q : p$ space.

The parameters N and λ^* were calibrated using results of K_0 test on a reconstituted sample (Smith et al. (1992), see Mašín (2007)). The shape of the state boundary surface was taken into account in the calculation of the parameter N from the position of the K_0 normal compression line in the $\ln(1+e) : \ln(p/p_r)$ space. The final sensitivity s_f is equal to one, as full destructuration is observed in K_0 compression experiments on natural Bothkennar clay (Smith et al., 1992). Because the set of stress probing tests published by Smith et al. (1992) does not include equivalent experiments on reconstituted soil, other parameters including the initial value of sensitivity were evaluated directly using stress probing data on natural Bothkennar clay by means of parametric studies.

The parameters of the hypoplastic and SMCC models and the initial values of state variables are summarised in Tabs. 1 and 2.

Predictions of the probing tests by the hypoplastic and SMCC models are shown in Figs. 8 and 9. The comparison resembles results of evaluation using Pisa clay data – the hypoplastic model reproduces well the measured behaviour, both inside the state boundary surface and on the surface. The SMCC model is capable of predicting correctly the large-strain behaviour, but in the pre-yield region it predicts incorrectly the elastic response with non-decreasing stiffness.

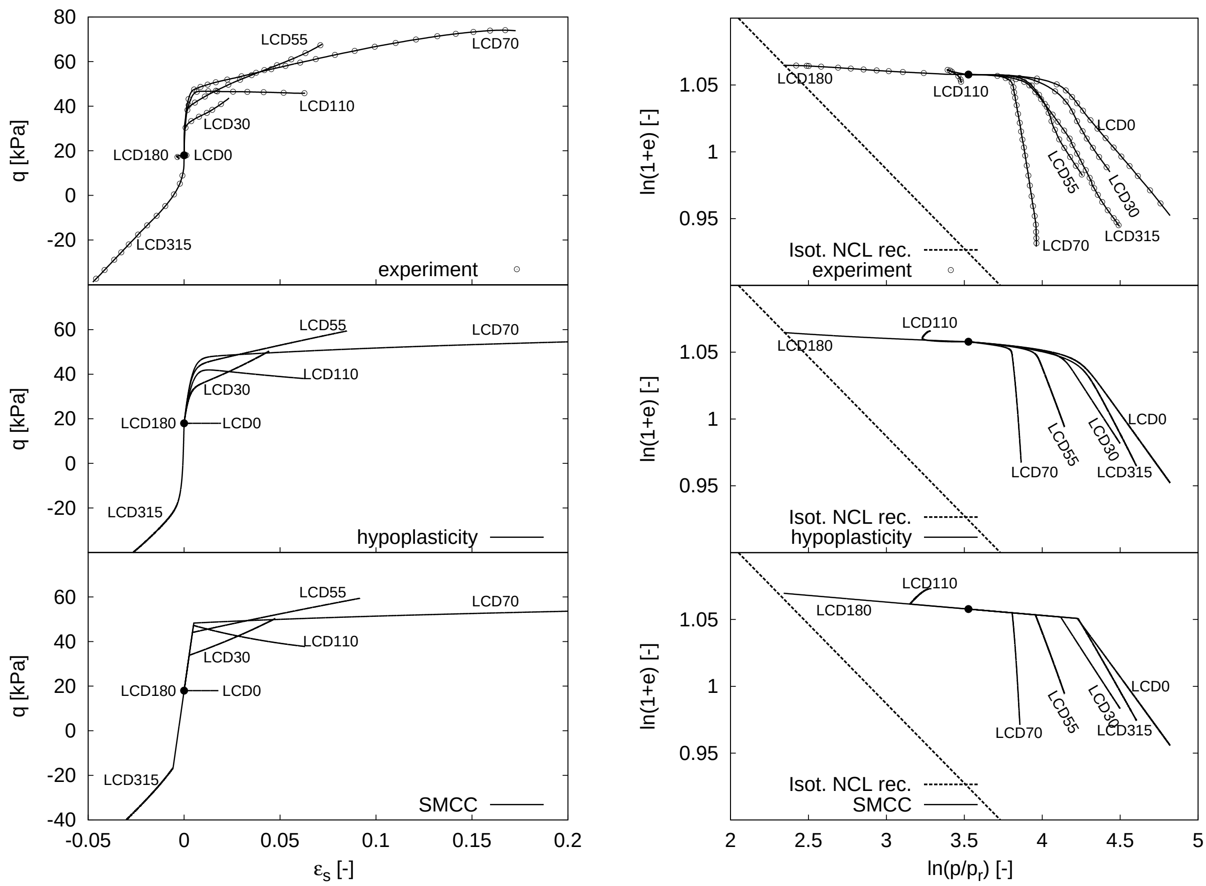


Figure 8: (a) ϵ_s vs. q curves and (b) $\ln(p/p_r)$ vs. $\ln(1+e)$ graphs from experiments on natural Bothkennar clay. Experimental data and predictions by the hypoplastic and SMCC models.

Several experiments on Bothkennar clay (LCD0, 55, 70 and 315) have been simulated by Baudet (2001) using a kinematic hardening model for structured clays by Baudet and Stallebrass (2004). Unlike the kinematic hardening model by Rouainia and Muir Wood (2000), whose predictions have been shown in the previous section, the model by Baudet and Stallebrass (2004) considers circular shape of the SBS in the octahedral plane and it does not take into account fabric anisotropy by using non-isotropic shape of the SBS. On the other hand, it introduces the third kinematic surface, which improves predictions in the small-strain range and enables us to model the effects of recent history. The model requires altogether 11 parameters plus the final sensitivity s_f. For details of the model formulation, calibration and simulation of tests on Bothkennar clay see Baudet (2001).

Predictions by the kinematic hardening model (from Baudet (2001)) are in Fig. 10 compared with predictions by the hypoplastic model and with experimental data. Both the models give

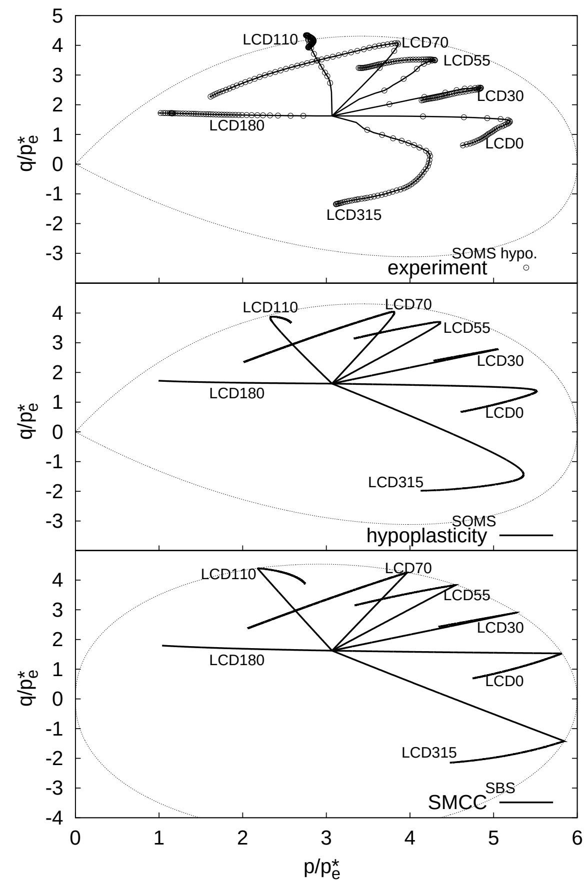


Figure 9: Normalised stress paths of experiments on natural Bothkennar clay. Experimental data and predictions by the hypoplastic and SMCC models.

qualitatively similar predictions with non-linear stress-strain response inside the SBS. The large strain response is for some tests predicted more accurately by the kinematic hardening model (particularly tests LCD70 and 315).

Summary and conclusions

Comparison of predictive capabilities of a recently proposed hypoplastic model for structured clays (Mašín, 2007), a simple elasto-plastic critical state model for structured clays (SMCC), and two advanced elasto-plastic kinematic hardening models for structured clays (Callisto et al. (2002) and Baudet and Stallebrass (2004)) has been presented in the paper.

The hypoplastic model and the SMCC model require the same number of material constants and state variables with similar physical interpretation, they can thus be regarded as equivalent from the point of view of practising engineer. The hypoplastic model, thanks to its capabilities of predicting non-linear soil behaviour inside the state boundary surface, different stiffness in different loading directions and smooth structure degradation process that takes place already inside the SBS, provides a clear qualitative and quantitative advance with respect to the SMCC model. Still, both models predict similar large-strain behaviour, which shows that advanced critical state soil mechanics theories can be treated successfully within the framework of the theory of hypoplasticity.

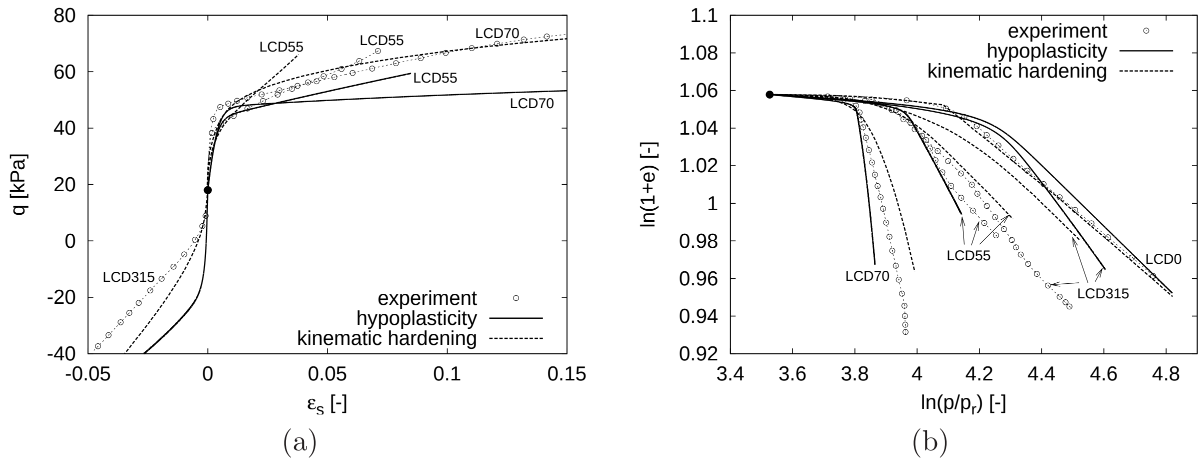


Figure 10: (a) ϵ_s vs. q curves and (b) $\ln(p/p_r)$ vs. $\ln(1+e)$ graphs of selected experiments on natural Bothkennar clay. Experimental data and predictions by the hypoplastic and kinematic hardening (Baudet, 2001) models.

The hypoplastic model gives qualitatively similar predictions as the advanced kinematic hardening elasto-plastic models. Both approaches predict non-linear stress-strain response also inside the state boundary surface. The kinematic hardening models provide in some cases more accurate predictions from the quantitative point of view. This improvement is, however, payed by larger number of material parameters and state variables and more problematic implementation into numerical codes.

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Appendix A

The mathematical formulation of the hypoplastic model for clays with meta-stable structure is summarised briefly in the following. The rate formulation of the hypoplastic model reads

$$\dot{\boldsymbol{\sigma}} = f_s \boldsymbol{\mathcal{L}} : \dot{\boldsymbol{\epsilon}} + f_s f_d \mathbf{N} \left\| \dot{\boldsymbol{\epsilon}} \right\| \tag{8}$$

The fourth-order tensor $\boldsymbol{\mathcal{L}}$ is a hypoelastic tensor given by

$$\boldsymbol{\mathcal{L}} = 3 \left(c_1 \boldsymbol{\mathcal{I}} + c_2 a^2 \hat{\boldsymbol{\sigma}} \otimes \hat{\boldsymbol{\sigma}} \right) \tag{9}$$

with the two scalar factors c_1 and c_2 introduced by Herle and Kolymbas (2004) and modified by Mašín (2005):

$$c_1 = \frac{2\left(3 + a^2 - 2^{\alpha} a\sqrt{3}\right)}{9rS_i} \qquad\qquad c_2 = 1 + (1 - c_1)\frac{3}{a^2} \tag{10}$$

where the scalars a and α are functions of the material parameters φ_c, λ^* and κ^*

$$a = \frac{\sqrt{3}\,(3 - \sin\varphi_c)}{2\sqrt{2}\sin\varphi_c} \qquad\qquad \alpha = \frac{1}{\ln 2}\ln\left[\frac{\lambda^* - \kappa^* S_i}{\lambda^* + \kappa^* S_i}\left(\frac{3 + a^2}{a\sqrt{3}}\right)\right] \tag{11}$$

and S_i is a factor calculated from model parameters k and s_f and a state variable sensitivity s:

$$S_i = \frac{s - k(s - s_f)}{s} \tag{12}$$

The second-order tensor $\mathbf{N}$ is given by Niemunis (2002)

$$\mathbf{N} = \boldsymbol{\mathcal{L}} : \left(Y\frac{\mathbf{m}}{\|\mathbf{m}\|}\right) \tag{13}$$

where the quantity Y determines the shape of the critical state locus in the stress space such that for $Y = 1$ it coincides with the Matsuoka and Nakai (1974) limit stress condition.

$$Y = \left(\frac{\sqrt{3}a}{3 + a^2} - 1\right)\frac{(I_1 I_2 + 9I_3)\left(1 - \sin^2\varphi_c\right)}{8I_3\sin^2\varphi_c} + \frac{\sqrt{3}a}{3 + a^2} \tag{14}$$

with the stress invariants

$$I_1 = \operatorname{tr}(\boldsymbol{\sigma}) \qquad\qquad I_2 = \frac{1}{2}\left[\boldsymbol{\sigma} : \boldsymbol{\sigma} - (I_1)^2\right] \qquad\qquad I_3 = \det(\boldsymbol{\sigma})$$

$\det(\boldsymbol{\sigma})$ is the determinant of $\boldsymbol{\sigma}$. The second-order tensor $\mathbf{m}$ has parallel in the flow rule in elasto-plasticity. It is calculated by

$$\mathbf{m} = -\frac{a}{F}\left[\hat{\boldsymbol{\sigma}} + \operatorname{dev}\hat{\boldsymbol{\sigma}} - \frac{\hat{\boldsymbol{\sigma}}}{3}\left(\frac{6\hat{\boldsymbol{\sigma}} : \hat{\boldsymbol{\sigma}} - 1}{(F/a)^2 + \hat{\boldsymbol{\sigma}} : \hat{\boldsymbol{\sigma}}}\right)\right] \tag{15}$$

with the factor F

$$F = \sqrt{\frac{1}{8}\tan^2\psi + \frac{2 - \tan^2\psi}{2 + \sqrt{2}\tan\psi\cos 3\theta}} - \frac{1}{2\sqrt{2}}\tan\psi \tag{16}$$

where

$$\tan\psi = \sqrt{3}\,\|\operatorname{dev}\hat{\boldsymbol{\sigma}}\| \qquad\qquad \cos 3\theta = -\sqrt{6}\frac{\operatorname{tr}\left(\operatorname{dev}\hat{\boldsymbol{\sigma}}\cdot\operatorname{dev}\hat{\boldsymbol{\sigma}}\cdot\operatorname{dev}\hat{\boldsymbol{\sigma}}\right)}{\left[\operatorname{dev}\hat{\boldsymbol{\sigma}} : \operatorname{dev}\hat{\boldsymbol{\sigma}}\right]^{3/2}} \tag{17}$$

The *barotropy* factor f_s introduces the influence of the mean stress level. The way of its derivation ensures that the hypoplastic model predicts correctly the isotropic normally compressed states.

$$f_s = S_i \frac{3p}{\lambda^*} \left(3 + a^2 - 2^\alpha a \sqrt{3}\right)^{-1} \tag{18}$$

The *pyknotropy* factor f_d incorporates the influence of the overconsolidation ratio. The critical state is characterised by $f_d = 1$ and the isotropic normally compressed state by $f_d = 2^\alpha$.

$$f_d = \left(\frac{2p}{sp_e^*}\right)^\alpha \qquad\qquad p_e^* = p_r \exp\left[\frac{N - \ln(1+e)}{\lambda^*}\right] \tag{19}$$

with the reference stress $p_r = 1$ kPa. Finally, evolution of the state variables e (void ratio) and s (sensitivity) is governed by

$$\dot{e} = -\left(1 + e\right)\dot{\epsilon}_v \tag{20}$$

$$\dot{s} = -\frac{k}{\lambda^*}(s - s_f)\sqrt{(\dot{\epsilon}_v)^2 + \frac{A}{1-A}\left(\dot{\epsilon}_s\right)^2} \tag{21}$$

$\dot{\epsilon}_v$ and $\dot{\epsilon}_s$ are rates of volumetric and shear strains respectively and A is a model parameter.

Appendix B

The appendix presents a complete mathematical formulation of the "Structured Modified Cam clay" (SMCC) model. The rate formulation of the model reads

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{\mathcal{D}}^e : (\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}^p) \tag{22}$$

The elastic stiffness matrix $\boldsymbol{\mathcal{D}}^e$ is calculated from the shear modulus G (constitutive parameter) and bulk modulus K, related to the parameter κ^* via

$$K = \frac{p}{\kappa^*} \tag{23}$$

by

$$\boldsymbol{\mathcal{D}}^e = \left(K - \frac{2}{3}G\right) \mathbf{1} \otimes \mathbf{1} + 2G\boldsymbol{\mathcal{I}} \tag{24}$$

Yield surface (f) is associated with the plastic potential (g) surface

$$f = g = q^2 + M^2 p\left(p - s^{ep} p_c^*\right) \tag{25}$$

M is the model parameter, s^{ep} (sensitivity) is the state variable and the quantity p_c^* is related to the state variable e (void ratio) through the equation

$$p_c^* = p_r \exp\left(\frac{N - \kappa^* \ln p - \ln\left(1+e\right)}{\lambda^* - \kappa^*}\right) \tag{26}$$

where p_r is the reference stress 1 kPa and N and λ^* are model parameters. Inside the yield surface ($f < 0$), $\dot{\boldsymbol{\epsilon}}^p = \mathbf{0}$. For stress states on the yield surface, the plastic strain rate is given by the following rule:

$$\dot{\boldsymbol{\epsilon}}^p = \frac{\langle \mathbf{m} : \boldsymbol{\mathcal{D}}^e : \dot{\boldsymbol{\epsilon}} \rangle}{H + \mathbf{m} : \boldsymbol{\mathcal{D}}^e : \mathbf{m}} \, \mathbf{m} \tag{27}$$

where the operator $\langle x \rangle := (x + |x|)/2$ denotes the positive part of any scalar function x, H is the plastic modulus calculated from the consistency condition

$$H = \frac{M^2 p p_c^*}{\lambda^* - \kappa^*} \left[s^{ep} \operatorname{tr}(\mathbf{m}) - k \left(s^{ep} - s_f^{ep} \right) \sqrt{\operatorname{tr}^2(\mathbf{m}) + \left(\frac{A}{1 - A} \right) \frac{2}{3} \operatorname{dev}(\mathbf{m}) : \operatorname{dev}(\mathbf{m})} \right] \tag{28}$$

and the tensor $\mathbf{m}$ is calculated by:

$$\mathbf{m} = \frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{M^2 (2p - s^{ep} p_c^*)}{3} \mathbf{1} + 3 \operatorname{dev}(\boldsymbol{\sigma}) \tag{29}$$

s_f^{ep} and k are model parameters. Evolution of state variables is governed by equations:

$$\dot{e} = -\left(1 + e\right) \dot{\epsilon}_v \tag{30}$$

$$\dot{s}^{ep} = -\frac{k}{\lambda^* - \kappa^*} (s^{ep} - s_f^{ep}) \sqrt{\left(\dot{\epsilon}_v^p\right)^2 + \frac{A}{1 - A} \left(\dot{\epsilon}_s^p\right)^2} \tag{31}$$

$\dot{\epsilon}_v^p$ and $\dot{\epsilon}_s^p$ are rates of plastic volumetric and shear strains respectively and A is a model parameter.