An effective stress based model for the dependency of a water retention curve on void ratio

D. Mašín
Charles University in Prague, Czech Republic

ABSTRACT: The paper presents a model for the dependency of a water retention curve (WRC) on void ratio. The approach is based on the effective stress principle for unsaturated soils and several underlying assumptions. The model describes the unsaturated soil behaviour along the main drying and wetting branches of WRC, it therefore does not incorporate the effects of hydraulic hysteresis. It leads to the dependency of a water retention curve (WRC) on void ratio, which does not require any material parameters apart from the parameters specifying WRC for the reference void ratio. Its predictive capabilities are demonstrated by comparing predictions with the experimental data on several different soils.

1 INTRODUCTION

Water retention curve (WRC), which quantifies the dependency of a degree of saturation S_r on suction s in unsaturated soils, is not unique for the soil of particular granulometry and mineralogy. WRC depends on suction path, it is thus different for wetting and drying processes. In addition, the main drying and wetting branches of WRC depend on the actual void ratio e.

The importance of considering the dependency of WRC on e has currently been well recognised and it has been incorporated into a number of recently proposed constitutive models for hydraulic behaviour of unsaturated soils, among other by Sun et al. (2008), Gallipoli et al. (2003), Nuth and Laloui (2008) and Wheeler et al. (2003). In these models, the dependency of WRC on e is controlled by an appropriately chosen empirical relationship controlled by additional model parameters.

In this paper, an alternative to these relationships is developed on the basis of several fundamental assumptions. No additional parameters are needed to describe the dependency of WRC on e, which simplifies the parameter determination procedure. Hydraulic hysteresis is not considered in the present derivations; the proposed model may be considered as suitable for description of the main wetting and drying branches of WRC. Nonetheless, it may easily be incorporated into a more advanced model predicting the hysteretic hydraulic behaviour.

More details on the proposed approach are given by Mašín (2010).

2 ADOPTED DESCRIPTIONS OF THE STRESS STATE AND WATER RETENTION CURVE

2.1 Stress state description

Central to the proposed approach is the adopted description of a stress state within unsaturated material. It now becomes generally accepted that two stress measures are needed for proper description of the stress state within unsaturated soil. A tensorial stress measure describing the averaged action of external forces and fluid pressures on the soil skeleton and a scalar stress measure quantifying the stiffening effect of water menisci on the skeleton.

The tensorial stress measure $\boldsymbol{\sigma}'$ may be in general written as

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \chi u_w \mathbf{1} - (1-\chi) u_a \mathbf{1} = \boldsymbol{\sigma}_{net} - \chi s \mathbf{1} \quad (1)$$

where $\boldsymbol{\sigma}$ is a total stress, u_a is the pore air pressure and u_w is the pore water pressure, $\boldsymbol{\sigma}_{net}$ is the net stress defined as $\boldsymbol{\sigma}_{net} = \boldsymbol{\sigma} - u_a \mathbf{1}$ and s is matric suction $s = -(u_a - u_w)$. χ is the Bishop factor. The incremental form of (1) is written as

$$\dot{\boldsymbol{\sigma}}' = \dot{\boldsymbol{\sigma}} - \psi \dot{u}_w \mathbf{1} - (1-\psi) \dot{u}_a = \boldsymbol{\sigma}_{net} - \psi \dot{s} \mathbf{1} \quad (2)$$

where

$$\psi = \frac{d(\chi s)}{ds} \quad (3)$$

Different formulations for the factor χ are adopted by different researchers. One of the possible approaches selects the tensorial variable such that it forms a frame within which the unsaturated soil behaviour can be uniquely described. The additional scalar stress variable is needed to control the size of the state boundary surface. Such a tensorial stress measure can then be seen as an equivalent to the effective stress in saturated materials.

Probably the most popular formulation for the factor χ is $\chi = S_r$. It was, however, derived from different theoretical considerations rather then from the actual observation of the unsaturated soil behaviour. Suitability of different stress measures was studied by Khalili and Khabbaz (1998) and Khalili et al. (2004). Based on experimental data on volume change and strength of overconsolidated soils they have shown that the most suitable expression for χ that would satisfy the aforementioned properties of effective stress in unsaturated soils does not appear to be the one with $\chi = S_r$. They proposed a formulation which does not involve the degree of saturation:

$$\chi = \begin{cases} 1 & \text{for} \quad s < s_e \\ \left(\frac{s_e}{s}\right)^{\gamma} & \text{for} \quad s \geq s_e \end{cases} \tag{4}$$

where s_e is the air entry value of suction (or air expulsion for wetting processes) and γ is an empirical coefficient. It was further shown that the best-fit value of the exponent $\gamma = 0.55$ is suitable to represent the behaviour of different soil types. γ can thus be considered as a material independent constant. s_e depends on the soil type and on the void ratio, though in the original formulation it is for simplicity considered as constant.

Eq. (4) leads using (3) to the following expression for the incremental effective stress factor ψ, which will be used as a basis for developments presented in this paper:

$$\psi = \begin{cases} 1 & \text{for} \quad s < s_e \\ (1-\gamma)\chi & \text{for} \quad s \geq s_e \end{cases} \tag{5}$$

2.2 Water retention curve

Unlike the exponent γ of the effective stress factor χ in (4), the WRC for a given void ratio depends significantly on the soil type and the granulometry. Many different mathematical relationships for the WRC with variable complexity are available throughout the literature. In this paper, a simple formulation formally similar to that by Brooks and Corey (1964) will be used:

$$S_r = \begin{cases} 1 & \text{for} \quad s < s_e \\ \left(\frac{s_e}{s}\right)^{\lambda_p} & \text{for} \quad s \geq s_e \end{cases} \tag{6}$$

valid for $S_r > S_{res}$, where S_{res} is the residual degree of saturation. s_e is the air-entry or air-expulsion suction as in Eq. (4). Similarly to the adopted expression for the effective stress factor χ, Eq. (6) neglects the effects of hydraulic hysteresis. It may therefore be considered as appropriate for representation of the main drying and wetting branches of the WRC. In (6), both factors s_e and λ_p depend on the soil type and on void ratio.

3 BASIC RELATIONS

Considering the existence of generalised elastic and plastic potentials defined in terms of effective stress for unsaturated soils, Loret and Khalili (2000) and Khalili et al. (2008) derived the following constitutive formulations for the pore water volume (V_w) and pore air volume (V_a) changes:

$$-\frac{\dot{V}_w}{V} = \psi\dot{\epsilon_v} - a_{11}\dot{u}_w - a_{12}\dot{u}_a \tag{7}$$

$$-\frac{\dot{V}_a}{V} = (1-\psi)\dot{\epsilon_v} - a_{21}\dot{u}_w - a_{22}\dot{u}_a \tag{8}$$

in which V is total volume of a soil element, $\dot{\epsilon}_v = -\dot{V}/V$ is the rate of soil skeleton volumetric strain, a_{ij} are material parameters and ψ is the effective stress rate factor from Eq. (5). The definition of S_r implies

$$\dot{V}_w = S_r\dot{V}_v + \dot{S}_r V_v \tag{9}$$

and therefore

$$\frac{\dot{V}_w}{V} = -S_r\dot{\epsilon}_v + n\dot{S}_r \tag{10}$$

$n = V_v/V$ is porosity. The degree of saturation S_r depends on suction s, void ratio and on the suction path. We may therefore write

$$\dot{S}_r = \frac{\partial S_r}{\partial s}\dot{s} + \frac{\partial S_r}{\partial \epsilon_v}\dot{\epsilon}_v \tag{11}$$

which can be substituted into (10):

$$\frac{\dot{V}_w}{V} = -\left(S_r - n\frac{\partial S_r}{\partial \epsilon_v}\right)\dot{\epsilon}_v + n\frac{\partial S_r}{\partial s}\dot{s} \tag{12}$$

Comparing (12) with (7) we have

$$\psi = S_r - n\frac{\partial S_r}{\partial \epsilon_v} = S_r + e\frac{\partial S_r}{\partial e} \tag{13}$$

where $e = n/(1-n)$ is void ratio. Substituting (13) into (11) yields the following general expression for the rate of the degree of saturation S_r:

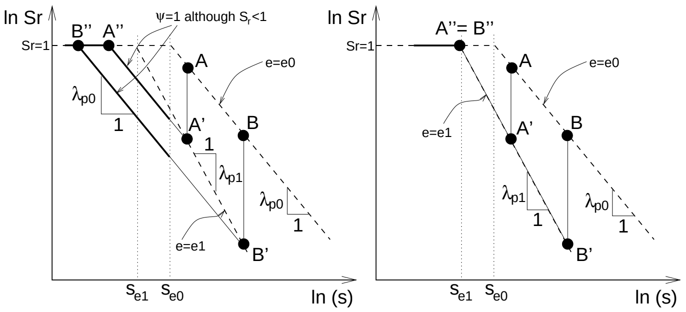
$$\dot{S}_r = \frac{\partial S_r}{\partial s}\dot{s} + \frac{\psi - S_r}{e}\dot{e} \tag{14}$$

The first term in (14) quantifies the WRC at constant void ratio, the second term evaluates the dependency of S_r on void ratio at constant suction. Eq. (14) was originally derived by Loret and Khalili (2000).

4 ADDITIONAL MOTIVATION FOR THE PRESENT WORK

Eq. (14) can be used to calculate the rate of S_r from rates of void ratio and suction. In the following it is shown that using the rate form for calculation of changes of S_r with s and e (Eq. (14)) in combination with the effective stress formulation of Eq. (4) with constant s_e, and WRC formulation of Eq (6) with constant λ_p, leads to incorrect results.

Consider two specimens of the same void ratio e_0 and air-expulsion suction s_{e0} at two different suction levels, shown in Fig. 1a. Both the two specimens lie on a single wetting branch of a water retention curve corresponding to e_0 (states A and B). The specimens are first subject to a void ratio increase at constant suction to a void ratio e_1 (paths A-A' and B-B'), followed by a suction decrease at constant void ratio e_1 up to a full saturation (paths A'-A" and B'-B").



(a) (b)

Figure 1. Theoretical experiment demonstrating motivation for the present work. (a) response by Eq. (14) without e-dependent s_e and λ_p; (b) response by the proposed model.

In the following it is *assumed* that the main drying and main wetting branches of WRC are only dependent on void ratio and independent on the stress history. Based on this assumption, the two paths A'-A" and B'-B" should lie on the same wetting branch of WRC corresponding to the void ratio e_1, as shown in Fig. 1b. Instead, direct application of Eq. (14) for calculation of changes of S_r with constant λ_p predicts that the two specimens follow different paths in S_r vs. s plane during suction decrease (Fig. 1a). The anticipated response may be obtained if void ratio dependent value of λ_p is considered. Note that Fig. 1b presumes $\lambda_{p1} > \lambda_{p0}$ for $e_1 > e_0$. This dependency is implied by derivations presented subsequently in Sec. 5.

In addition to the problems described above, application of the effective stress principle (4) with constant air-expulsion value s_{e0}, with the factor ψ defined as in Eq. (5), imposes incorrectly $\psi = 1$ for all states with $s < s_{e0}$, therefore also for states with $S_r < 1$ (Fig. 1a). This problem may be overcome in two ways. First, e-dependency of s_e may be considered. A suitable relationship is presented in Sec. 5. Second, the effective stress factor χ from Eq. (4) may be, considering the adopted formulation for WRC (6), equivalently expressed in terms of S_r

$$\chi = S_r^{(\gamma/\lambda_p)} \tag{15}$$

which eliminates s_e from the effective stress equation.

Eq. (15) may be seen as a link between the effective stress concept by Khalili and Khabbaz (1998) from Eq. (4) and the theoretically derived expression $\chi = S_r$. As will be shown further, studies of the behaviour of different soils showed that $(\gamma/\lambda_p) > 1$. The exponent γ/λ_p may thus be seen as an empirically revealed manifestation of simplifying assumptions adopted in the theoretical approaches leading to the derivation of $\chi = S_r$. Eq. (15) is indeed valid for the main wetting and drying branches of WRC, provided that the double-logarithmic expression for WRC (6) is considered as appropriate. This does not directly imply its applicability for S_r states on the hydraulic scanning curve.

5 QUANTIFICATION OF THE DEPENDENCY OF WRC ON VOID RATIO

In this section, solution for the dependency of WRC on e is presented. The formulas are applicable for unsaturated states, where $s \geq s_e$ and therefore $\psi = (1-\gamma)\chi$. The derivations are detailed in Mašín (2010). It has been shown that the rate of the suction at air entry/expulsion s_e can be calculated by

$$\dot{s}_e = -\frac{\gamma s_e}{e \lambda_{psu}} \dot{e} \tag{16}$$

with

$$\lambda_{psu} = \frac{\gamma}{\ln \chi_{0su}} \ln \left[\left(\chi_{0su}^{\frac{\lambda_{p0}}{\gamma}} - \chi_{0su} \right) \left(\frac{e}{e_0} \right)^{(\gamma-1)} + \chi_{0su} \right] \tag{17}$$

where $\chi_{0su} = (s_{e0}/s_e)^{\gamma}$. s_{e0} and λ_{p0} are values of s_e and λ_p corresponding to the reference void ratio e_0. Dependency of λ_p on void ratio and suction is given by

$$\lambda_p = \frac{\gamma}{\ln \chi_0} \ln \left[\left(\chi_0^{\frac{\lambda_{p0}}{\gamma}} - \chi_0 \right) \left(\frac{e}{e_0} \right)^{(\gamma-1)} + \chi_0 \right] \tag{18}$$

with $\chi_0 = (s_{e0}/s)^{\gamma}$.

Knowledge of s_e from (18) and λ_p from (17) may be used to calculate the value of S_r for given void ratio and suction using Eq. (6). In fact, Eqs. (16-18) describe a state surface in the s vs. e vs. S_r space, which is depicted in Fig. 2 for Pearl clay parameters from Tab. 1. Constant-void-ratio cross-sections through this surface represent water retention curves, shown in Fig. 3a. Figs. 2 and 3a also demonstrate how increasing void ratio leads to the decrease of the s_e value. Figure 3b shows the influence of e and s on the slope λ_p of the WRC.

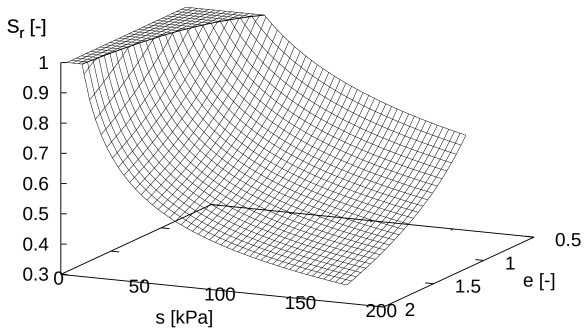


Figure 2. Predicted state surface in the s vs. e vs. S_r space.

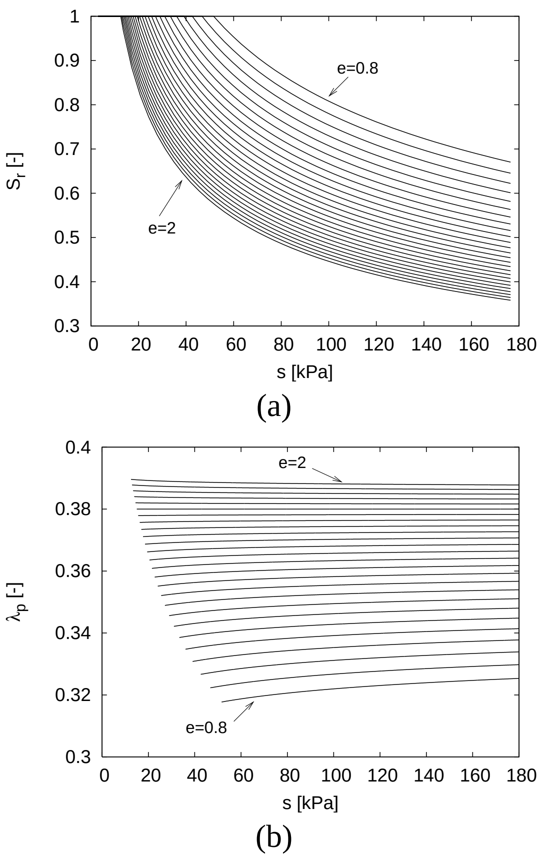


Figure 3. (a) The dependency of WRC on void ratio; (b) dependency of the slope λ_p of the WRC on s and e for reference values $\lambda_{p0} = 0.38$, $s_{e0} = 15$ kPa and $e_0 = 1.75$.

Two ways may be followed in the case the rate form of S_r is required (Eq. (14)). The derivative $\partial S_r/\partial s$ may either be found numerically from (6), or it can be substituted by $-\lambda_p S_r/s$ if the dependency of λ_p on s is neglected. This dependency is not significant for void ratios not significantly different as compared to the reference void ratio e_0 (see Fig. 3b).

Note that the model includes $\chi = S_r$ as a special case. In this case $\lambda_{p0} = \gamma$ and Eq. (18) simplifies to $\lambda_p = \lambda_{p0}$ (i.e., λ_p is independent of suction and of void ratio). The Equation (16) may then be integrated analytically, with resulting $s_e = s_{e0}e_0/e$.

6 DETERMINATION OF PARAMETERS

The model requires three material parameters. Parameters s_{e0} and λ_{p0} may be found directly by a bi-linear representation of the water retention curve in the $\ln S_r$ vs. $\ln(s/s_{e0})$ plane, as shown in Fig. 4. Parameter e_0 is void ratio corresponding to the approximated WRC. Obviously, this approach to model calibration assumes that the void ratio changes due to suction changes during the measurement of WRC are negligible. In the case the variation of e cannot be neglected, the calibration of λ_{p0} requires coupling of a proposed hydraulic model with some suitable mechanical model for unsaturated soils, evaluation of e for different suction states and comparison of the experimental WRC with an appropriate cross-section through the S_r:s:e state surface.

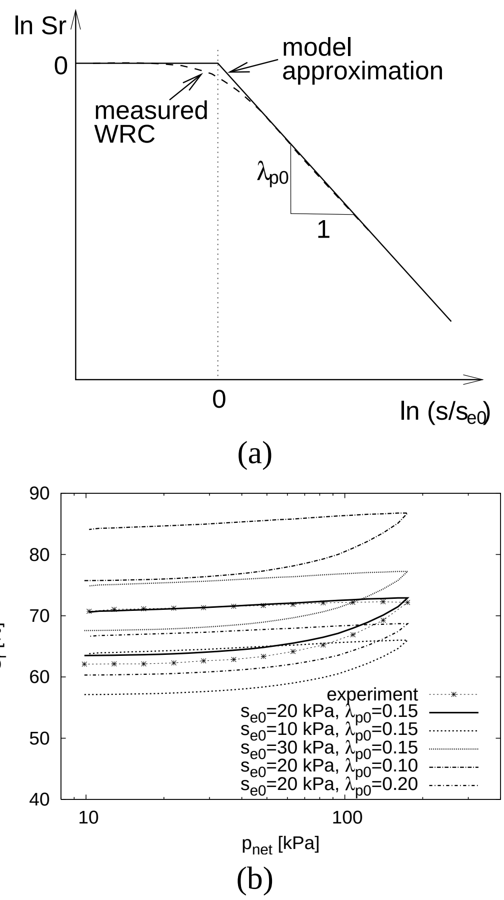


Figure 4. (a) Direct determination of parameters s_{e0}, λ_{p0} and e_0 by a bi-linear representation of the WRC; (b) calibration by the trial-and-error procedure using S_r vs. s vs. e data (experimental data by Sharma 1998).

Alternatively, the parameters may be found by a trial-and-error procedure using S_r vs. s vs. e data coming from laboratory experiments. In principle, any experiment with monotonous path may be used for this purpose. For example compression or shear tests at constant suction, wetting/drying tests at constant net stress, constant water content experiments, etc. Note that the value of e_0 may be selected arbitrarily to be in the range of reasonable void ratios for the given soil. The calibration using the trial-and-error procedure is demonstrated in Fig. 4b, which shows p_{net} vs. S_r diagram of constant suction isotropic loading-unloading test on bentonite/kaolin mixture by Sharma (1998) and predictions with the proposed model with $e_0 = 1.2$ and variable s_{e0} and λ_{p0}. Note that only one test is shown in Fig. 4b for clarity. Proper calibration using the trial-and-error procedure requires considering more tests with variable suction and/or void ratio.

7 COMPARISON WITH EXPERIMENTAL DATA

The proposed dependency of WRC on void ratio has been evaluated with respect to a wide range of different soils by Mašín (2010). All predictions were obtained using Eqs. (16-18). The predicted degree of saturation S_r was calculated directly from the experimentally measured e and s without a need to couple the present hydraulic model with a mechanical constitutive model for partly saturated soils. Parameters obtained are summarised in Tab. 1.

It is interesting to note that for all studied soils the value of λ_{p0} is significantly lower than $\gamma = 0.55$, demonstrating empirically that the exponent γ/λ_p from Eq. (15) takes values significantly higher than unity.

Table 1. Parameters of the proposed model calibrated on the basis of different experimental data (Mašín 2009).

soil	s_{e0} [kPa]	λ_{p0} [-]	e_0 [-]
Pearl clay	15	0.38	1.75
HPF quartz silt	3	0.18	0.7
Speswhite kaolin	65	0.3	1.4
bentonite/kaolin mix.	20	0.15	1.2

In the following, two examples of predictive capabilities of the proposed approach will be given.

7.1 HPF quartz silt

Jotisankasa et al. (2007) performed a set of constant water content oedometric tests with monitored suction on a mixture of 70 % silt of HPF type (consisting mainly of angular quartz grains), 10% kaolin and 20% London clay. The authors made available the WRC measured by standard filter paper technique at zero vertical stress, which allows us to calibrate the proposed model using the direct approach, demonstrated in Fig. 4a. Figure 5 shows the wetting branch of the WRC starting from compacted state at $s = 1000$ kPa and void ratio $e_0 = 0.7$. Figure 5 also shows the WRC predicted by the proposed model with $e_0 = 0.7$, $s_{e0} = 3$ kPa and variable λ_{p0}. The model curve for $\lambda_{p0} = 0.18$ reproduces well the experimental data, indicating suitability of Eq. (6) for the present purpose.

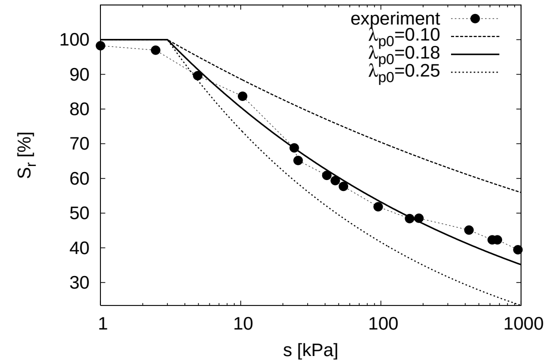


Figure 5. Wetting branch of WRC of HPF type quartz silt, experimental data by Jotisankasa et. al (2007), direct calibration of the proposed model with $e_0 = 0.7$, $s_{e0} = 3$ kPa and variable λ_{p0}.

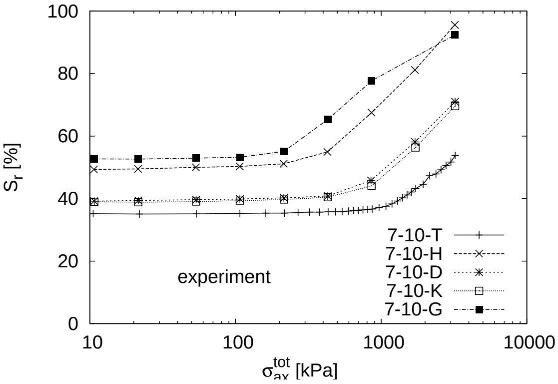
The oedometric tests by Jotisankasa et al. (2007) were performed at constant water content conditions, therefore both suction and void ratio varied during compression. Figures 6a shows the observed dependency of S_r on vertical stress for five different tests. The predictions are shown in Fig. 6b. The model is in a good match with the experiments, although tests with variable void ratio were not involved in model calibration.

7.2 Speswhite kaolin

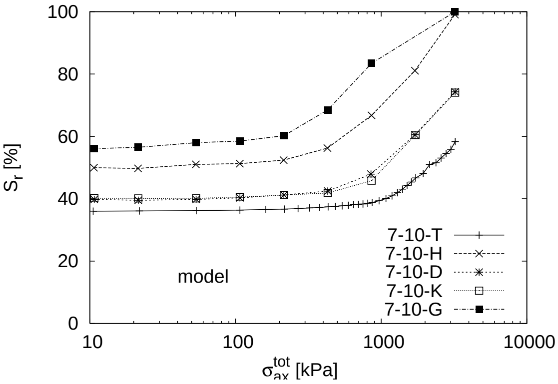
Tarantino and De Col (2008) studied the behaviour of Speswhite kaolin under static compaction at seven different water contents with continuous measurement of suction. The experimental results are in Fig. 7a shown in terms of S_r vs. s plots for different water contents. Figure 7b shows predictions by the proposed formulation with parameters calibrated by means of the trial-and-error procedure (described in Sec. 6). The proposed state surface represents well the measured behaviour for virgin loading. As expected, the model is due to the absence of hydraulic hysteresis less successful in predicting the behaviour upon unloading-reloading cycles.

8 CONCLUSIONS

Based on several underlying assumption including existence of generalised elastic and plastic potentials defined in terms of effective stresses for unsaturated soils, a new model for the dependency of WRC on void ratio has been formulated. The model does not require any model parameters apart from parameters specifying WRC for the reference void ratio. Good match between observed and predicted behaviour in-



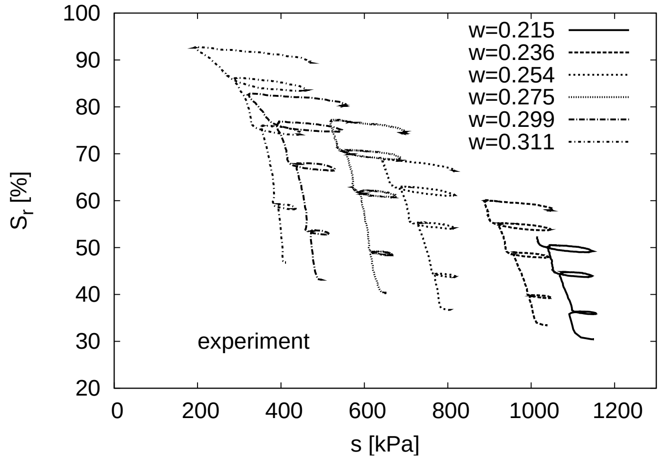
(a)



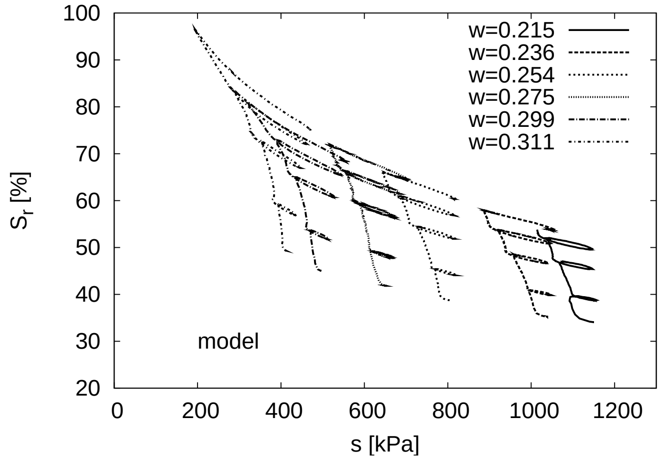
(b)

Figure 6. Results of suction-monitored oedometric tests at constant water content by Jotisankasa et. al (2007) (a), compared with model predictions of S_r using parameters calibrated directly from WRC (b).

directly supports the applicability of the effective stress concept for unsaturated soils.



(a)



(b)

Figure 7. (a) Results of compaction tests on Speswhite kaolin by Tarantino and De Col (2008) plotted in S_r vs. s graphs for constant water contents. (b) predicted results.

ACKNOWLEDGMENT

The author wish to thank to Prof. Nasser Khalili for valuable discussions on the subject and to Dr. Alessandro Tarantino for providing experimental data on Speswhite kaolin. Financial support by the research grants GACR 205/08/0732 and MSM 0021620855 is greatly appreciated.

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